Acoustic scattering from elastic cylinders and spheres: surface waves (Watson transform) and transmitted waves

Diffusion acoustique par des cylindres et des sphères élastiques : ondes de surface (transformation de Watson) et ondes transmises

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Electromagnetic waves: Establishment of the relation between resonance scattering theory and the Singularity Expansion Method (SEM). Theory of radio-wave propagation in the earth-ionosphere waveguide.
Nuclear theory: Basic development of coherent bremsstrahlung theory (editor of recent book). Development of a model of collective nuclear multipole vibrations, with application to electron scattering (author of two books). Photo-pion reactions. Collective model of ion-ion scattering. Neutrino reactions (editor of book).

SUMMARY

The classical normal-mode series of acoustic scattering from solid elastic cylinders and spheres is reformulated in terms of the S-function as developed in nuclear scattering theory. It is then subjected to the Watson transformation, which permits an evaluation of the scattering amplitude at its poles ("Regge poles") and saddle points in the complex mode-number plane. The saddle point contributions are obtained after expanding the amplitude in a Debye series, and correspond to a reflected wave and to transmitted dilatational and shear waves that undergo internal reflections and mode conversions. The theory of these waves was experimentally verified by Quentin *et al.* The pole residues furnish circumferential (surface, creeping) waves which are of both Franz type (propagating externally), and of elastic type (Rayleigh and Whispering Gallery waves, propagating internally). The theory of these waves was experimentally verified by Ripoche *et al.*

KEY WORDS

Normal mode series, Watson transformation, Regge poles, saddle points, reflected and transmitted waves, circumferential waves, surface waves, creeping waves, Franz type, Rayleigh type, Whispering Gallery type.

RÉSUMÉ

On reformule la série classique des modes normaux, qui décrit la diffusion acoustique par des cylindres et des sphères solides élastiques en l'exprimant par la fonction S, comme elle est développée dans la théorie de la diffusion nucléaire. Elle est alors soumise à la transformation de Watson, ce qui permet une évaluation de l'amplitude de diffusion à ses pôles (« pôles de Regge ») et à ses cols dans le plan complexe du numéro de mode. Les contributions des cols sont obtenues après le développement de l'amplitude en « série de Debye », et elles correspondent à une onde réfléchie et à des ondes transmises de type dilatation et cisaillement, soumises à des réflexions internes et des conversions de mode. La théorie de ces ondes a été vérifiée expérimentalement par Quentin et al. *Les résidus des pôles fournissent des ondes circonférentielles (ondes de surface, « Creeping waves », ou du type de Franz, se propageant sur le côté externe de l'interface), ou du type élastique (ondes de Rayleigh et de galerie à écho, se propageant sur le côté interne de l'interface). La théorie de ces ondes a été vérifiée expérimentalement par Ripoche* et al.

MOTS CLÉS

Série des modes normaux, transformation de Watson, pôles de Regge, cols, ondes réfléchies et transmises, ondes circonférentielles, ondes de surface, ondes du type Franz, type Rayleigh, type galerie à écho.

CONTENTS

1. Introduction

The first comprehensive theory of sound scattering from elastic cylinders and spheres was pressented by Faran [1] in 1951; further extensive work on spheres is due to Hickling [2]. In their analysis, the accoustic field is represented as a Rayleigh-type series summed over normal modes labeled by the mode number n, the modes having been obtained by the separation of variables in cylindrical or spherical coordinates, respectively. Although this solution can be numerically evaluated for obtaining, e. g., the backscattering cross section as a function of frequency, it does not lend itself to any physical understanding of the scattering process.

Such an understanding was provided, however, by the work of Franz [3, 4] on the scattering of electromagnetic waves from perfect conductors, this being the analogue of acoustic scattering from impenetrable (soft or rigid) objects. His application of the Watson transformation [5] provided a separation of the scattered field into two parts, one being a wave which in the limit of large values of ka (with k the acoustic wave number, a the radius of the cylinder or sphere) corresponds to geometrical reflection from the apex of the object, and the other representing a series of surface waves (called "creeping waves" by Franz) which propagate circumferentially around the scatterer, thus in effect making up the diffraction phenomenon.

This approach was carried further, and applied to the case of penetrable cylinders [6-8] and spheres [9-11]. Here, interior fields are present, and the following separation of the total field can be made: (a) a "geometric" part which in the large-ka limit corresponds to both a specular reflection from the apex, and to rays refracted into the interior of the scatterer, which then re-emerge into the exterior fluid either immediately, or after a series of multiple internal reflections. For an elastic object, mode conversion into rays of shear type may occur during these refractions or internal reflections; (b) a surface wave part in which circumferential waves propagate around the scatterer both externally (of "Franz type", similar to those for impenetrable objects), and internally (of "Rayleigh" and "Whispering Gallery" type, the former corresponding to the Rayleigh wave in the limit of a flat surface).

Comprehensive experimental studies have been performed which demonstrated qualitatively and quantitatively the correctness of this theory. At large frequencies ($ka \gtrsim 100$) where the geometrical fields dominate, studies of the reflection of short acoustic pulses from elastic cylinders, spheres and shells were carried out by Quentin *et al.* [12-16]. They verified by a measurement of both the pulse arrival times and their amplitudes, the existence and the predicted properties of transmitted waves, including their multiple reflections and mode conversions. The surface waves on elastic cylinders have been studies extensively by Ripoche *et al.* [17-19] at lower values of $ka (\lesssim 50)$, and the predicted dispersion and absorption curves of both Franz-type and elastic-type (internal) surface waves were confirmed experimentally.

2. Theory of the scattering process

The total acoustic fieeld in the presence of an infinite elastic cylinder is given in cylindrical coordinates (r, φ, z) by:

$$(2.1c)\quad p=\sum_{n=0}^{\infty}(2-\delta_{n0})\,i^n \times\{J_n(kr)+T_n H_n^{(1)}(kr)\}\cos n\varphi$$

(corresponding to normal incidence of sound; for oblique incidence, *see* [20]); for a sphere, it is in spherical coordinates (r, θ, φ):

$$(2.1s)\quad p=\sum_{n=0}^{\infty}(2n+1)\,i^n \times\{j_n(kr)+T_n h_n^{(1)}(kr)\}\,P_n(\cos\theta).$$

The incident pressure field, given by the first terms in equations (2.1), is here normalized to unity, and T_n is the partial-wave scattering amplitude ("T-function") in the normal-mode or Rayleigh series of equations (2. 1). It is customary in nuclear physics to rewrite the total amplitude in the form:

$$(2.2c)\quad p=\frac{1}{2}\sum_{n=0}^{\infty}(2-\delta_{n0})\,i^n \times\{H_n^{(2)}(kr)+S_n H_n^{(1)}(kr)\}\cos n\varphi,$$

or:

$$(2.2s)\quad p=\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\,i^n \times\{h_n^{(2)}(kr)+S_n h_n^{(1)}(kr)\}P_n(\cos\theta),$$

where:

$$(2.3)\qquad S_n=2T_n+1,\qquad T_n=\frac{1}{2}(S_n-1)$$

gives the relation between T_n and the "S-function" S_n. Satisfying the boundary conditions at $r=a$ leads to:

$$(2.4)\qquad S_n=S_n^{(s)}\frac{F_n-Z_n^{(2)}}{F_n-Z_n^{(1)}},$$

with:

$$(2.5c)\qquad Z_n^{(i)}=x\,H_n^{(i)\prime}(x)/H_n^{(i)}(x),$$

$$(2.5s)\qquad Z_n^{(i)}=x\,h_n^{(i)\prime}(x)/h_n^{(i)}(x),$$

$i=1$, 2 and $x=ka$, where:

$$(2.6c)\qquad S_n^{(s)}=-H_n^{(2)}(x)/H_n^{(1)}(x)$$

$$(2.6s)\qquad S_n^{(s)}=-h_n^{(2)}(x)/h_n^{(1)}(x),$$

are the S-functions for a soft scatterer (for rigid scatterers, $H_n^{(i)\prime}$ or $h_n^{(i)\prime}$ appear). The quantities F_n are given in the literature [1]; they are proportional to ρ/ρ_0, the density ratio of the fluid and the scatterer, so that $F\to\infty$ for a soft, and $F\to 0$ for a rigid object.

The Watson transformation [3-5] consists in rewriting the normal-mode sum as a contour integral in the complex n-plane:

$$(2.7c)\quad p=\frac{i}{2}P\int_C\frac{d\nu}{\sin\pi\nu}e^{-i\nu\pi/2} \times\{H_\nu^{(2)}(kr)+S_\nu H_\nu^{(1)}(kr)\}\cos\nu\varphi,$$

$$(2.7s)\quad p=\frac{1}{2i}\int_C\frac{\lambda\,d\lambda}{\cos\pi\lambda}\{h_{\lambda-(1/2)}^{(2)}(kr) +S_{\lambda-(1/2)}h_{\lambda-(1/2)}^{(1)}(kr)\}\,P_{\lambda-(1/2)}(-\cos\theta),$$

where C tightly surrounds the positive real axis in the complex ν or λ plane, passing through $\nu=0$ (P=principal value) but to the right of $\lambda=0$. The Imai transformation:

$$(2.8c)\quad \cos\nu\varphi=e^{i\nu\pi}\cos\nu(\varphi-\pi)-ie^{i\nu(\pi-\varphi)}\sin\pi\nu,$$

$$(2.8s)\quad P_{\lambda-(1/2)}(-\cos\theta)=e^{-i\pi(\lambda-1/2)} \times P_{\lambda-(1/2)}(\cos\theta)-2i\cos\pi\lambda\,Q_{\lambda-(1/2)}^{(-)}(\cos\theta),$$

where

$$(2.9)\quad Q_\mu^{(\pm)}(\cos\theta)=\frac{1}{2}\left\{P_\mu(\cos\theta)\mp\frac{2i}{\pi}Q_\mu(\cos\theta)\right\}$$

defines $Q_\mu^{(\pm)}$ in terms of the Legendre function of the second kind, Q_μ, splits equations (2. 7) into two portions (note that the terms with $H_\nu^{(2)}$ or $h_{\lambda-(1/2)}^{(2)}$ integrate to zero). The first one, containing $\cos\nu(\varphi-\pi)$ or $P_{\lambda-1/2}(\cos\theta)$, can be evaluated at the poles of the S-functions S_ν in the ν-plane ("Regge poles"), given by the zeros of $F_\nu-z_\nu^{(1)}$ and denoted by $\nu=\nu_l\,(l=1, 2, 3\ldots$ labeling their multiplicity):

$$(2.10c)\quad p_{cw}=\sum_{l=1}^{\infty}\frac{\pi}{\sin\pi\nu_l}e^{i\pi\nu_l/2} \times S_{\nu_l}^{(R)}H_{\nu_l}^{(1)}(kr)\cos\nu_l(\varphi-\pi),$$

$$(2.10s)\quad p_{cw}=\sum_{l=1}^{\infty}\frac{\pi(\nu_l+1/2)}{\sin\pi\nu_l}e^{-i\pi\nu_l} \times S_{\nu_l}^{(R)}h_{\nu_l}^{(1)}(kr)\,P_{\nu_l}(\cos\theta),$$

where $S_{\nu_l}^{(R)}$ is the residue of S_ν at $\nu=\nu_l$. The exponential form of $\cos\nu_l(\varphi-\pi)$, and the asymptotic form:

$$(2.11)\quad P_{\nu_l}(\cos\theta)\sim\left\{\frac{2}{\pi(\nu_l+1/2)\sin\theta}\right\}^{1/2} \times\sum_{\varepsilon=\pm1}e^{i\varepsilon(\nu_l+(1/2))\theta-i\varepsilon(\pi/4)},$$

demonstrate that equations (2. 10) represent circumferential waves p_{cw} with propagation constant ν_l for the cylinder, $\nu_l+1/2$ for the sphere. From this, one finds the phase velocities:

$$(2.12\,a)\quad c_l^c=\frac{x}{\operatorname{Re}\nu_l}c,\qquad c_l^s=\frac{x}{\operatorname{Re}\nu_l+(1/2)}c,$$

and the attenuation angles (amplitude $\exp -\varphi/\varphi_l$):

$$(2.12\,b)\qquad \varphi_l=\theta_l=\frac{1}{\operatorname{Im}\nu_l},$$

of the surface waves, both the external and the internal ones.

The second part of equations (2.7) is:

$$(2.13\,c)\quad p_{gw}=\frac{1}{2}\int_c e^{i\nu(\pi/2-\varphi)}\,S_\nu H_\nu^{(1)}(kr)\,d\nu,$$

$$(2.13\,s)\quad p_{gw}=-\int_c \lambda S_{\lambda-(1/2)}\,h_{\lambda-(1/2)}^{(1)}(kr)\times Q_{\lambda-(1/2)}^{(-)}(\cos\theta)\,d\lambda.$$

These integrals can be evaluated at the saddle points of the integrand [3, 4, 8, 9] and are then seen to represent geometrical waves p_{gw}, either reflected from the apex of the scatterer or undergoing internal transmissions without and with multiple internal reflections, including mode conversions for an elastic scatterer. This is obtained from an expansion of S_ν in equations (2.13) into a Debye series, as was done for an elastic cylinder by Brill and Überall [8], and for an elastic sphere by Gérard [10], but which for the simpler case of fluid scatterers [8, 9] becomes, e. g. for cylinders:

$$(2.14\,a)\quad S_\nu=S_\nu^{(s)}\left\{R_{12}-\frac{H_\nu^{(1)}(\beta x)}{H_\nu^{(2)}(\beta x)}T_{12}T_{21}\times\sum_{k=1}^{\infty}\left(\frac{H_\nu^{(1)}(\beta x)}{H_\nu^{(2)}(\beta x)}R_{21}\right)^{k-1}\right\},$$

where $\beta=c/c_0$, the sound velocity ratio of fluid and scatterer. This contains an external reflection coefficient:

$$(2.14\,b)\quad R_{12}=\left\{\frac{H_\nu^{(2)\prime}(x)}{H_\nu^{(2)}(x)}-N\frac{H_\nu^{(2)\prime}(\beta x)}{H_\nu^{(2)}(\beta x)}\right\}U^{-1}$$

and an internal one,

$$(2.14\,c)\quad R_{21}=-\left\{\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)}-N\frac{H_\nu^{(1)\prime}(\beta x)}{H_\nu^{(1)}(\beta x)}\right\}U^{-1},$$

where:

$$(2.14\,d)\quad U=\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)}-N\frac{H_\nu^{(2)\prime}(\beta x)}{H_\nu^{(2)}(\beta x)},$$

as well as the transmission coefficients:

$$(2.14\,e)\quad T_{12}=1-R_{12},\qquad T_{21}=1+R_{21},$$

into and out of the object, respectively; also, $N=\beta\rho/\rho_0$. For the sphere, one replaces $H_\nu^{(i)}\to h_\nu^{(i)}$ in equations (2.14). The individual terms in equations (2.14), evaluated at their saddle points in equations (2.13), furnish the geometrical rays corresponding to no (R_{12}) or k internal traversals.

3. Experimental results

The surface waves form resonating standing waves around the scatterer for $\operatorname{Re}\nu_l=n$, where $n\,(n\,(1/2))$ of their wavelengths span the cylinder (sphere; here, a

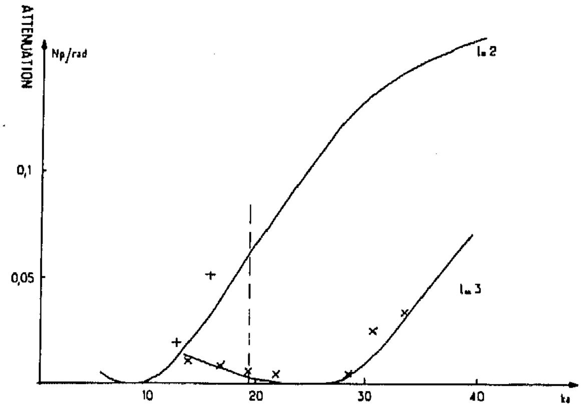

Fig. 3.1. — **Attenuation of the Whispering Gallery waves $l=2$, 3. Solid curves: theory [7]; crosses: experiment [19].**

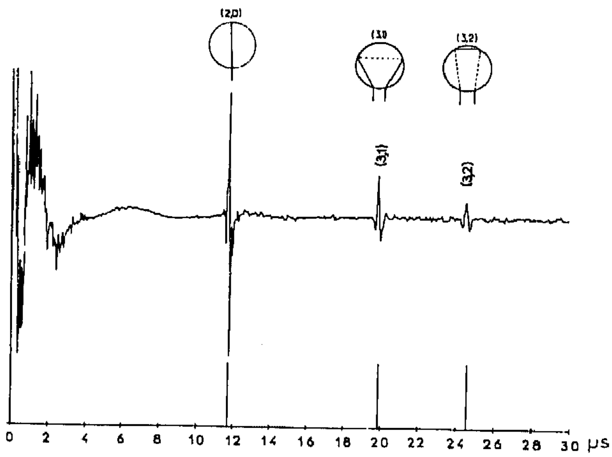

Fig. 3.2. — **Experimental [12] and theoretical [9] arrival times of short sound pulses traversing a lucite cylinder (solid rays: dilatational; broken rays: shear).**

$\lambda/4$ phase jump occurs at each of their two convergence points [21]). By observing these resonances on cylinders, Ripoche *et al.* [17-19] have verified the surface waves, and determined their phase velocities c_l^c and attenuations φ_l, equations (2.12). Figure 3.1 shows their results (crosses) for the attenuation of Whispering Gallery waves (internal, $l=2,3$) as compared to theory [7] (curves).

For the transmitted waves in cylinders and spheres, experiments [12, 13] give excellent agreement with the theoretical arrival times and amplitudes [8]. Figure 3.2 shows measured and calculated arrival times of short acoustic pulses backscattered from a lucite cylinder (the inserts show the type of traversals).

4. Conclusion

The theory of sound scattering from elastic cylinders and spheres, as based on the Watson transformation, gives an accurate account of the physics of the diffraction phenomenon. Its various aspects, i. e. the existence and properties of (resonating) external and internal circumferential waves, and of geometrically reflected and internally transmitted rays, have all been verified experimentally. Analogous studies with scatterers of more general shapes are now called for.

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