SIGGRAPH 2010

“Spectral Mesh Processing”

Bruno Lévy and Richard Hao Zhang

Spectral Mesh Processing Applications 2/2

Bruno Lévy

INRIA - ALICE

Overview

I. **1D parameterization**

II. Surface quadrangulation

III. Surface parameterization

IV. Surface characterization

Green function

Heat kernel

1D surface parameterization
Graph Laplacian

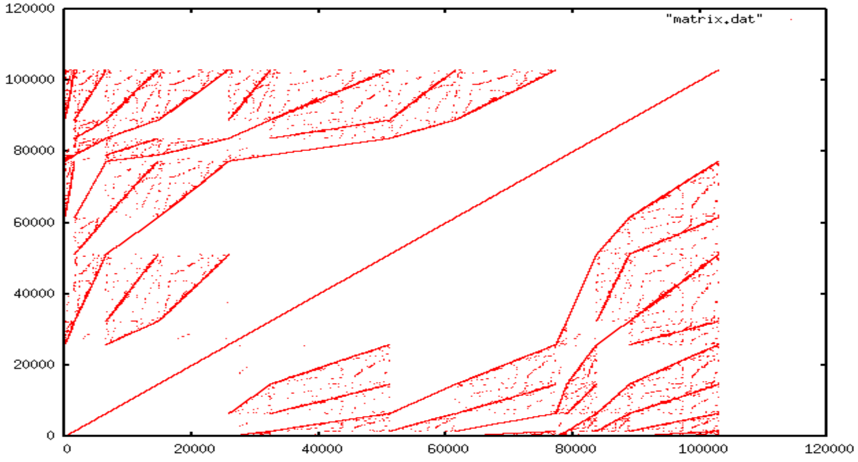
$a_{i,j} = w_{i,j} > 0$ if (i,j) is an edge

$a_{i,i} = -\sum a_{i,j}$

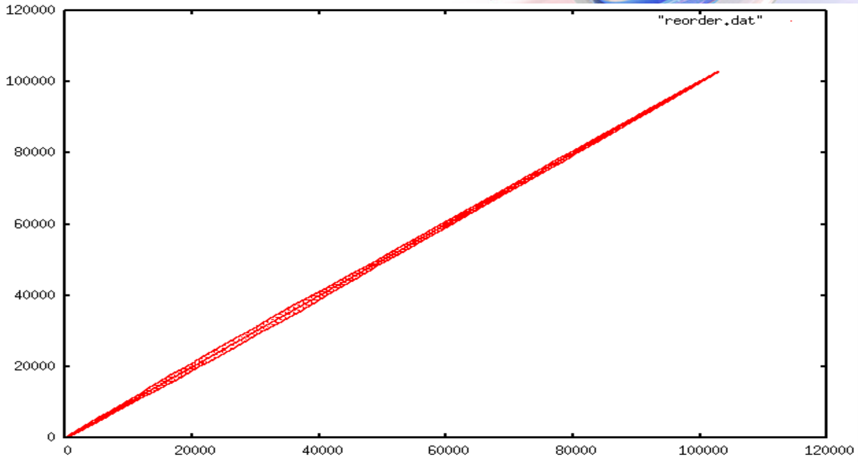
(1,1 … 1) is an eigenvector assoc. with 0

The second eigenvector is interresting
[Fiedler 73, 75]

1D surface parameterization
Fiedler vector



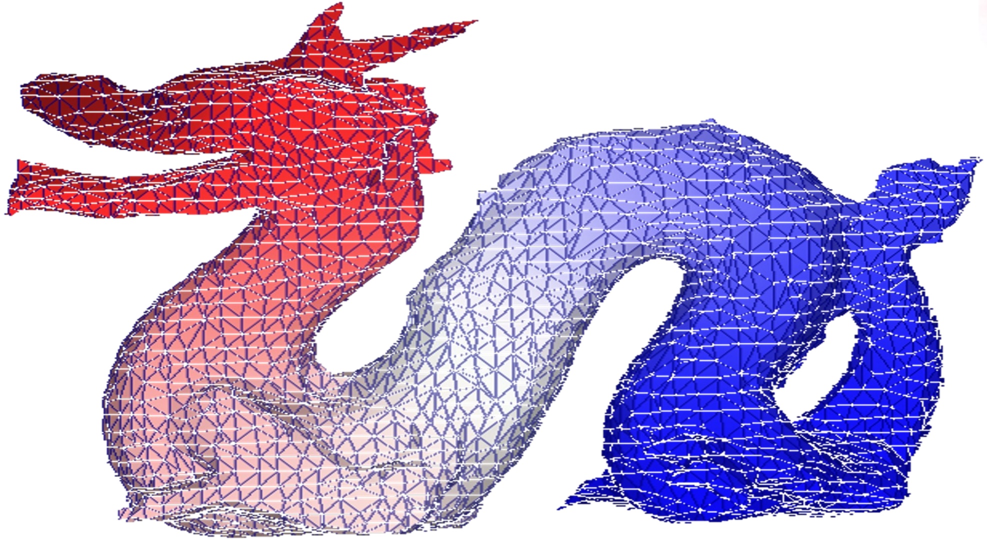
FEM matrix,
Non-zero entries



Reorder with
Fiedler vector

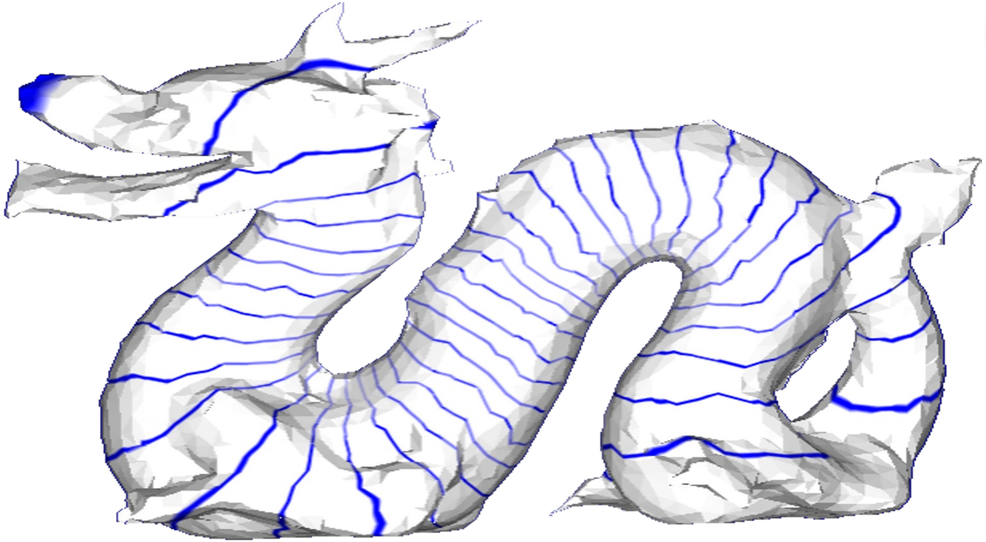
1D surface parameterization

Fiedler vector



Streaming meshes
[Isenburg & Lindstrom]

1D surface parameterization
Fiedler vector

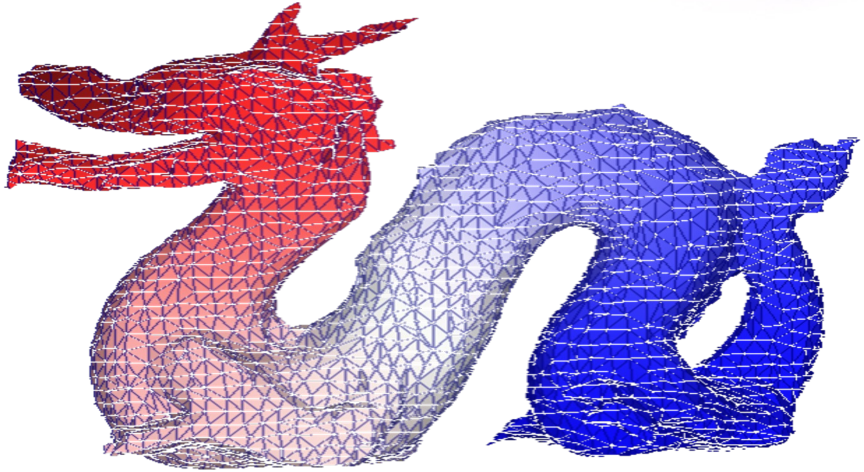


Streaming meshes
[Isenburg & Lindstrom]

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$



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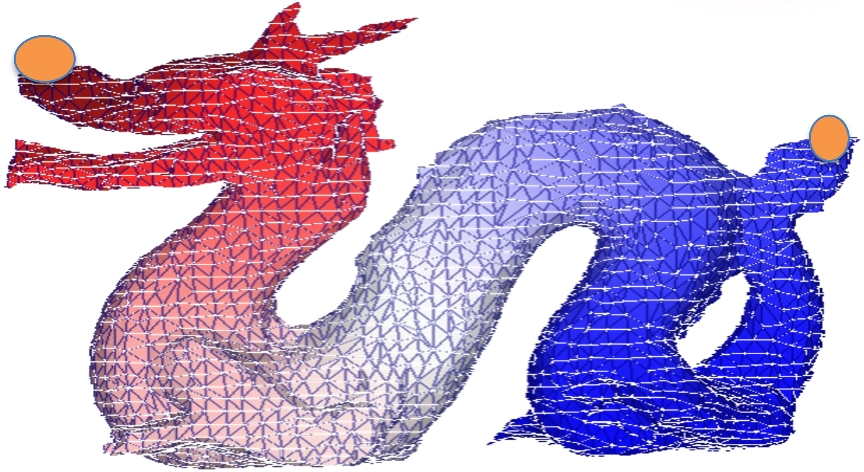

How to avoid trivial solution ?
Constrained vertices ?

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$ subject to $\sum u_i = 0$

Global constraints are more elegant !

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$ subject to

$$\sum u_i = 0$$

$$\frac{1}{2} \sum u_i^2 = 1$$

Global constraints are more elegant !

We need also to constrain the second mementum

1D surface parameterization
Fiedler vector

$$F(u) = \Sigma\, w_{ij}\, (u_i - u_j)^2$$

Minimize $F(u) = \frac{1}{2}\, u^t A\, u$ subject to $\sum u_i = 0$, $\frac{1}{2} \sum u_i^2 = 1$

$$L(u) = \tfrac{1}{2}\, u^t A\, u - \lambda_1\, u^t\, \mathbf{1} - \lambda_2\, \tfrac{1}{2}\, (u^t u - 1)$$

$$\nabla_u L = A\, u - \lambda_1\, \mathbf{1} - \lambda_2\, u$$
$$\nabla_{\lambda 1} L = u^t\, \mathbf{1}$$
$$\nabla_{\lambda 2} L = \tfrac{1}{2}(u^t\, u - 1)$$

u = eigenvector of A

$\lambda_1 = 0$

λ_2 = eigenvalue

1D surface parameterization
Fiedler vector

Rem: Fiedler vector is also a minimizer of the Rayleigh quotient

$$R(A,x) = \frac{x^t A x}{x^t x}$$

The other eigenvectors x_i are the solutions of :

minimize $R(A,x_i)$ subject to $x_i^t x_j = 0$ for $j < i$

Overview

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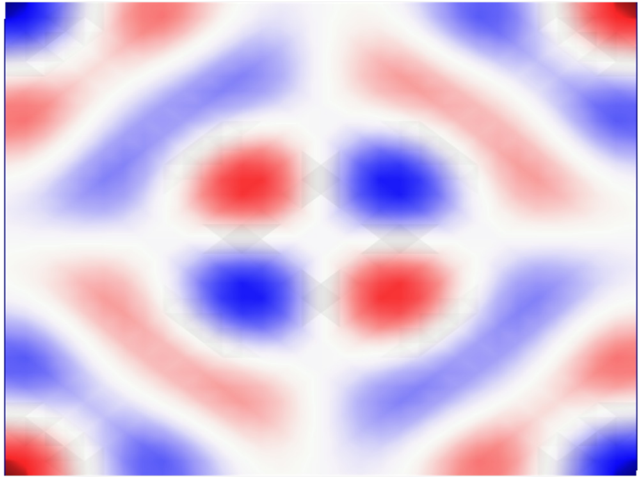
IV. Surface characterization

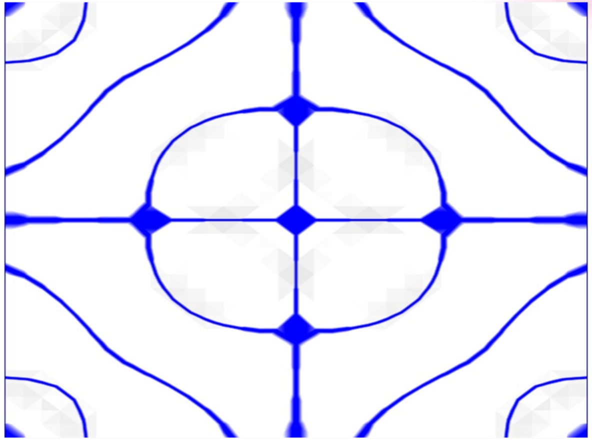
  Green function

  Heat kernel

Surface quadrangulation

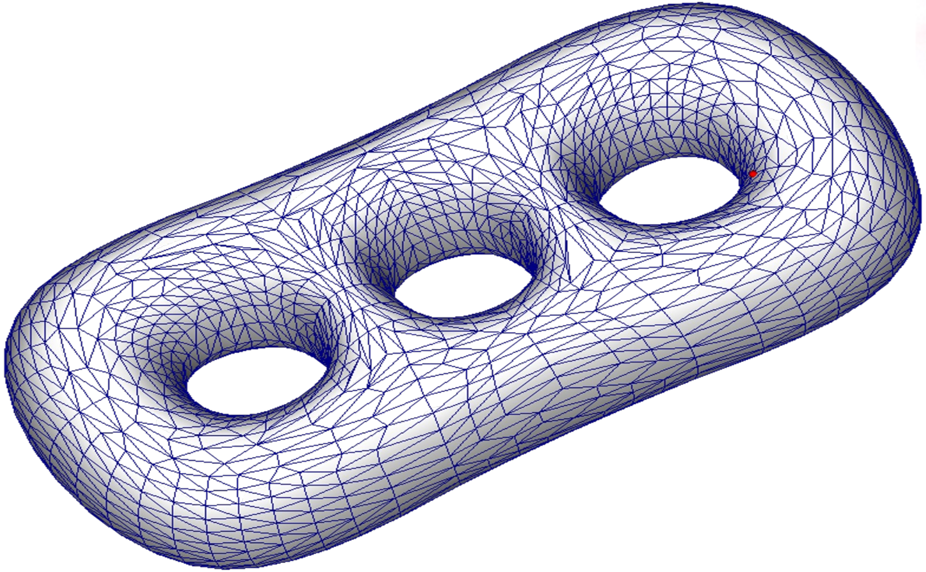
Nodal sets are sets of curves intersecting at constant angles



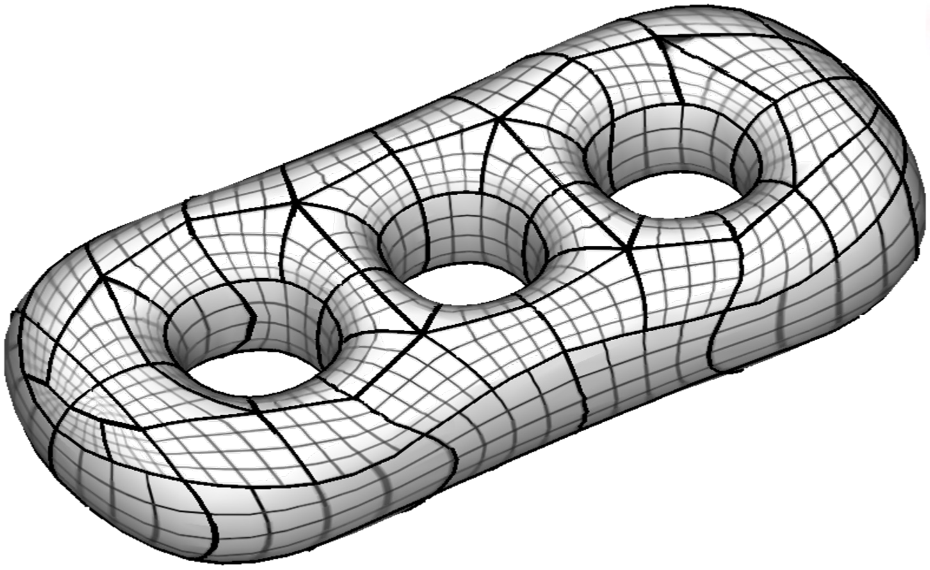


The N-th eigenfunction has at most N eigendomains

Surface quadrangulation

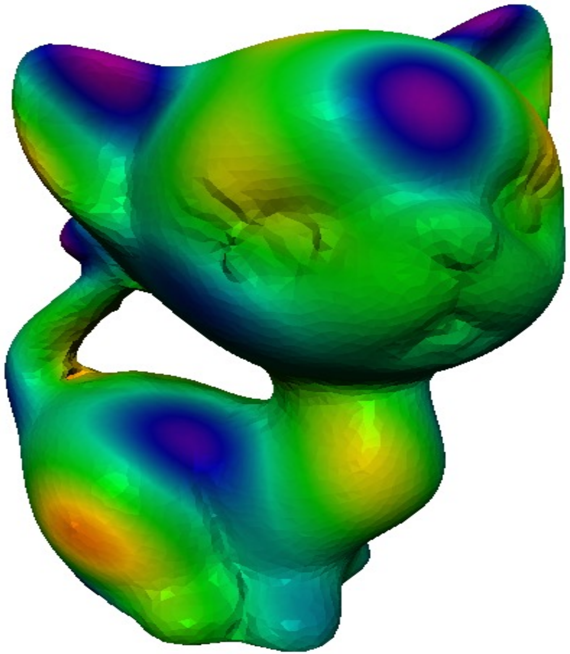


Surface quadrangulation

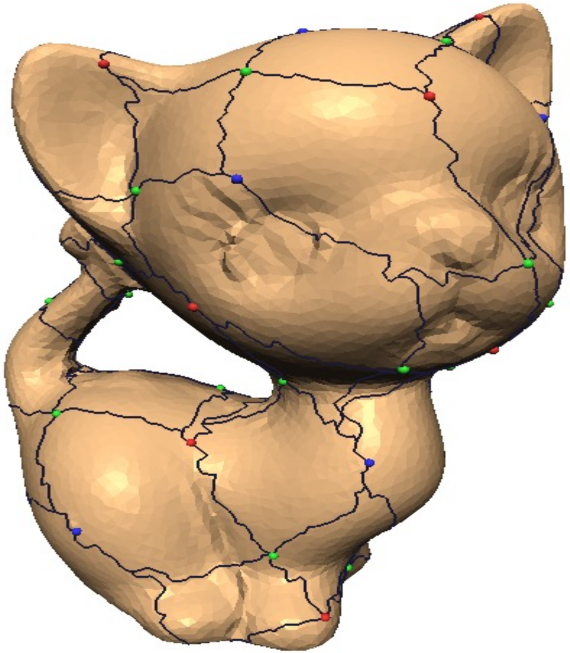


[L 2006], [Vallet & L 2006]

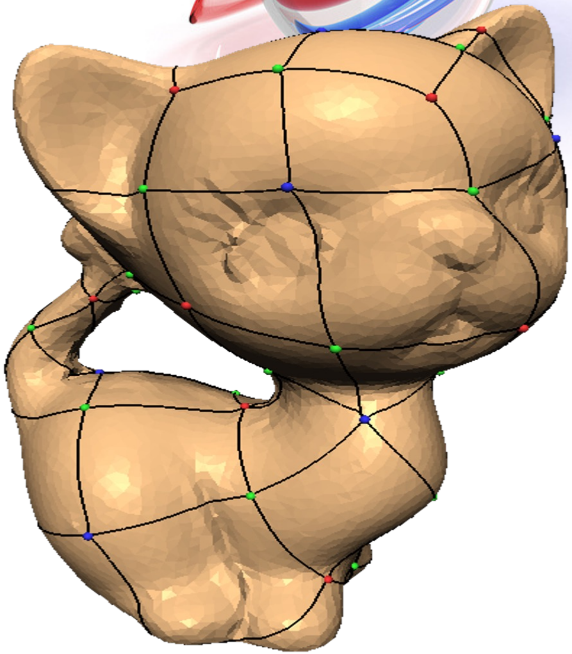
Surface quadrangulation



One eigenfunction



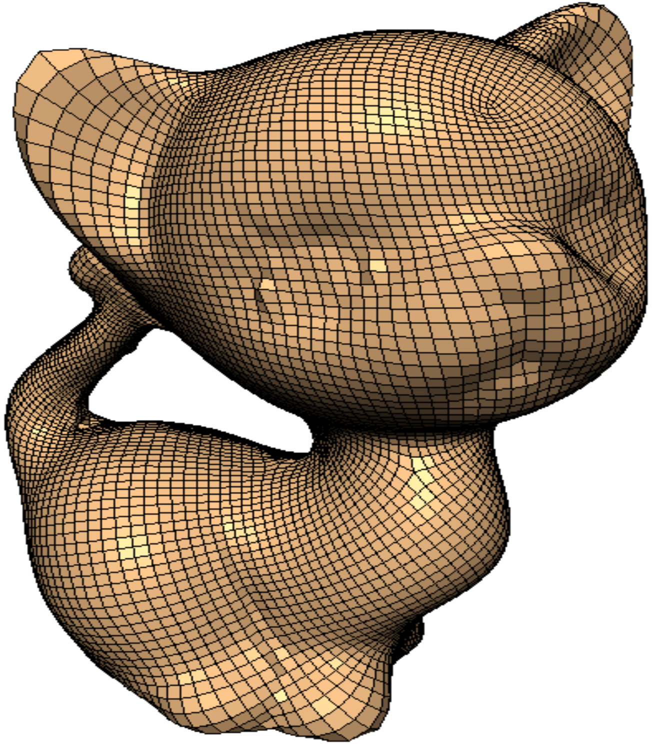
Morse complex

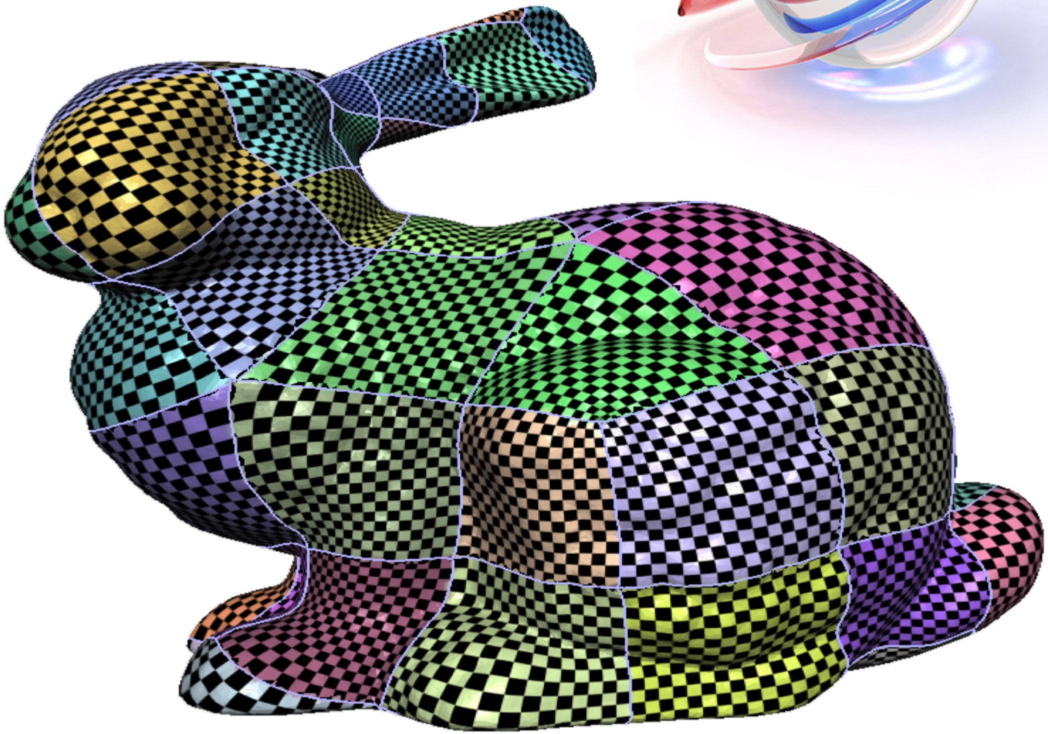


Filtered morse complex

[Dong and Garland 2006]

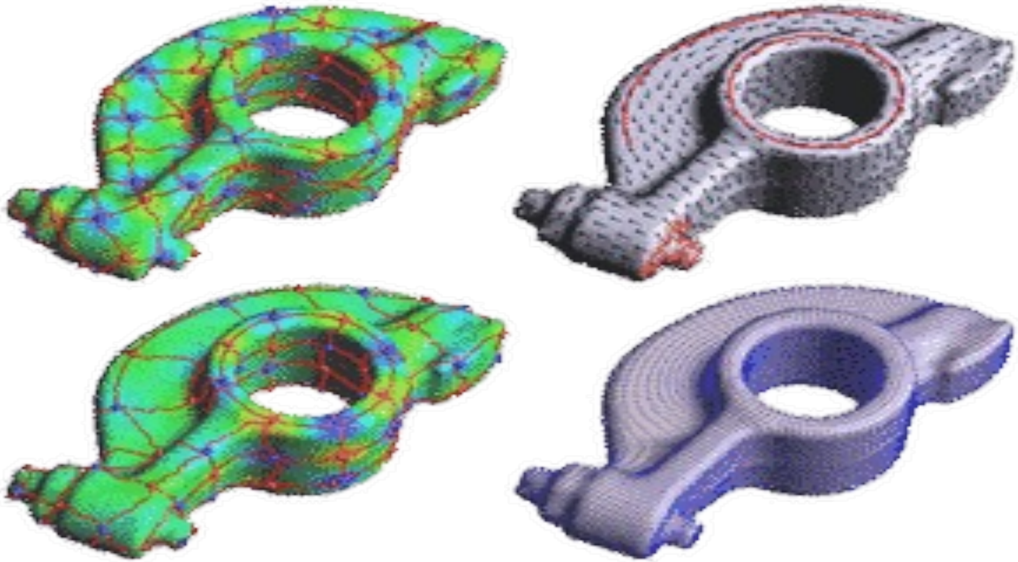
Surface quadrangulation





Reparameterization of the quads

Surface quadrangulation



Improvement in [Huang, Zhang, Ma, Liu, Kobbelt and Bao 2008], takes a guidance vector field into account.

Overview

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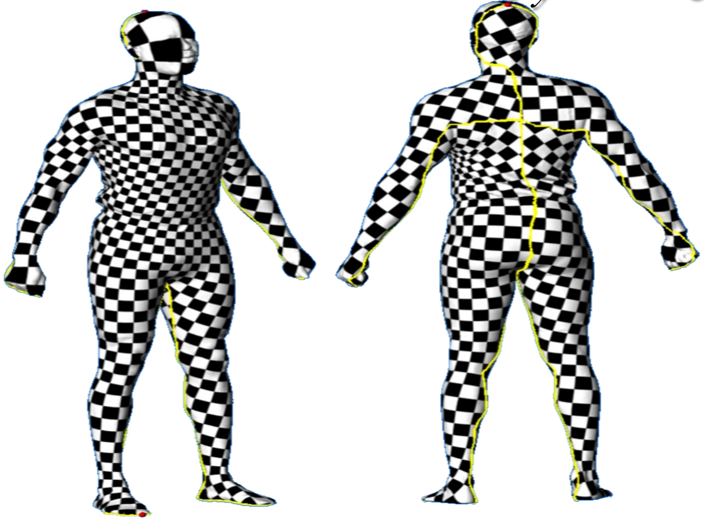
- Heat kernel

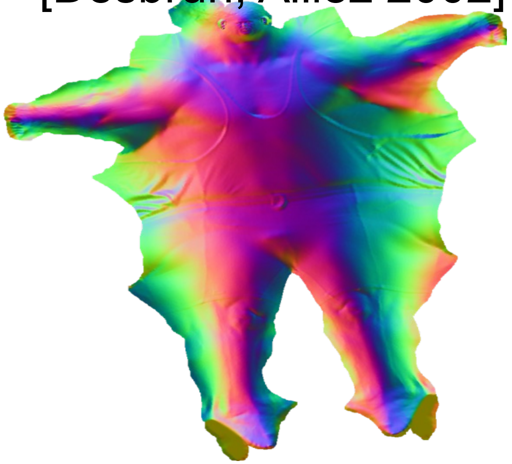
Surface parameterization

Minimize $$\sum_{T} \left\| \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} - \begin{bmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{bmatrix} \right\|^2$$

Discrete conformal mapping:

[L, Petitjean, Ray, Maillot 2002]
[Desbrun, Alliez 2002]





Surface parameterization

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Discrete conformal mapping:

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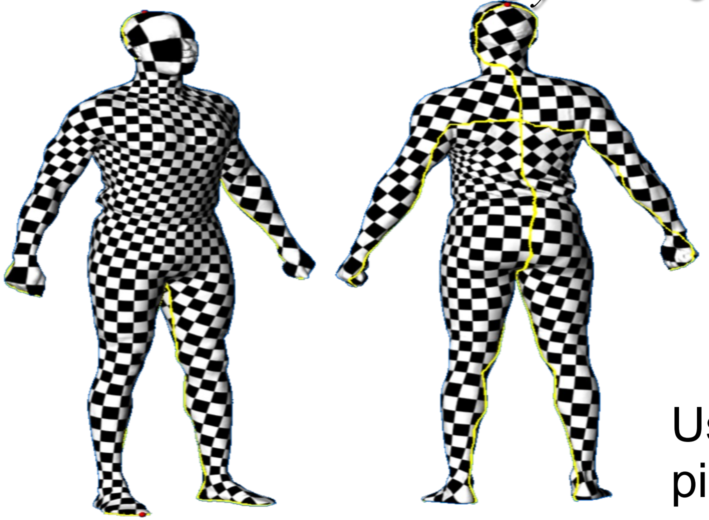
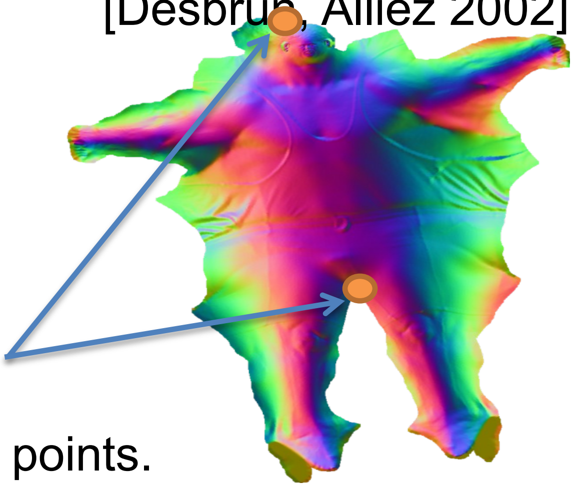
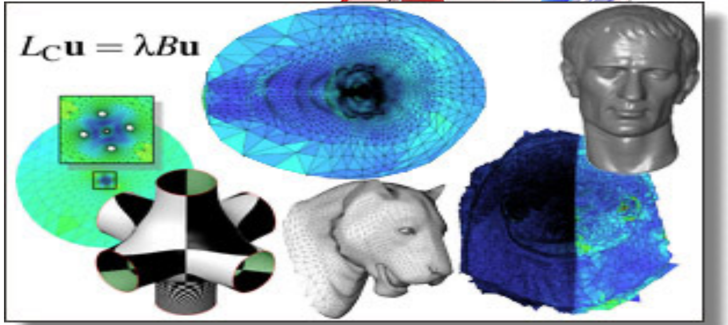




Uses pinned points.

Surface parameterization

[Muellen, Tong, Alliez, Desbrun 2008]



Use Fiedler vector,
i.e. the minimizer of $R(A,x) = x^t A x / x^t x$
that is orthogonal to the trivial constant solution

Implementation:
(1) assemble the matrix of the discrete conformal parameterization
(2) compute its eigenvector associated with the first non-zero eigenvalue

See http://alice.loria.fr/WIKI/ Graphite tutorials – Manifold Harmonics

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Green function

Heat kernel

Surface characterization

Green Function

Solving Poisson equation: $\Delta f = g$

$f = \int G(x,y) f(y)\, dy$

Where G: Green function is defined by: $\Delta G(x,y) = \delta(x-y)$

δ : dirac

Surface characterization

Green Function

Solving Poisson equation: $\Delta f = g$

$$f = \int G(x,y) f(y)\, dy$$

Where G: Green function is defined by: $\Delta G(x,y) = \delta(x-y)$

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Proof:

$$\int \Delta\, G(x,y) g(y)\, dy = \int \delta(x-y) g(y) dy = g(x) = \Delta\, f(x)$$

Surface characterization

Green Function

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$$\Delta\, f(x) = g(x) = \int \Delta\, G(x,y) g(y) dy = \Delta \left(\int G(x,y) g(y) dy \right)$$

Surface characterization

Green Function

Solving Poisson equation: $\Delta f = g$

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$$\Delta\, f(x) \;=\; g(x) = \int \Delta\, G(x,y)g(y)dy \;=\; \Delta \left(\int G(x,y)g(y)dy\right)$$

$$f(x) = \int G(x,y)g(y)dy$$

Surface characterization

Green Function

How to compute G ? G is defined by: $\Delta G(x,y) = \delta(x-y)$

δ : dirac

$\delta(x-y) = \sum \phi_i(x)\, \phi_i(y)$ (completeness of the eigen decomposition)

Surface characterization

Green Function

How to compute G ? G is defined by: $\Delta G(x,y) = \delta(x-y)$

δ : dirac

$\delta(x-y) = \sum \phi_i(x)\,\phi_i(y)$ (completeness of the eigen decomposition)

Using $G(x,y) = \dfrac{\sum \phi_i(x)\,\phi_i(y)}{\lambda_i}$ Works !

$$\left(\ \Delta G(x,y) = \sum \phi_i(x)\,\phi_i(y) = \delta(x-y)\ \right)$$

Note: Convergence of G series needs to be proved (complicated)

Surface characterization

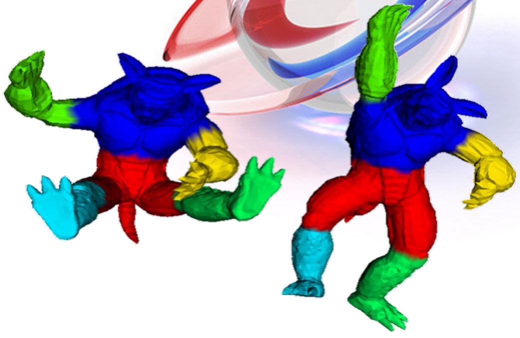
Green Function

Summary:

$$G(x,y) = \frac{\sum \phi_i(x)\, \phi_i(y)}{\lambda_i}$$

Solution Poisson equation: $\Delta f = g$

$$f = \int G(x,y) f(y)\, dy$$



Pose-invariant embedding

Connection with GPS embedding [Rustamov 2007]

$$GPS(x) = [\phi_1(x)/\sqrt{\lambda_1}\ ,\ \phi_2(x)/\sqrt{\lambda_2}\ ,\ \ldots\ ,\ \phi_i(x)/\sqrt{\lambda_i}\ ,\ \ldots\ ]$$

$$G(x,y) = GPS(x) * GPS(y)$$

Overview

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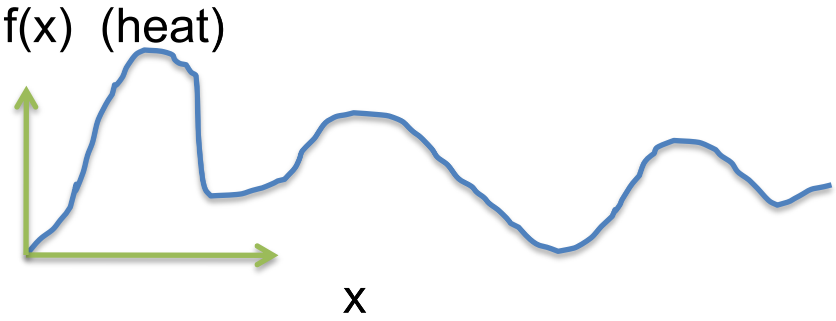
- **Heat kernel**

Surface characterization

Heat equation

The heat equation: $\frac{\partial f}{\partial t} = -\Delta f$

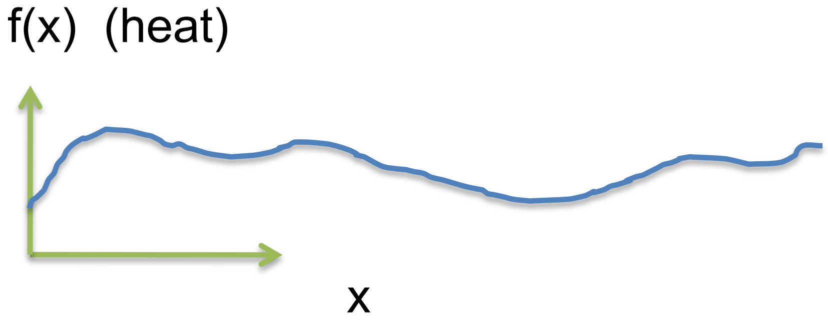


t = 0

Surface characterization

Heat equation

The heat equation: $\frac{\partial f}{\partial t} = -\Delta f$

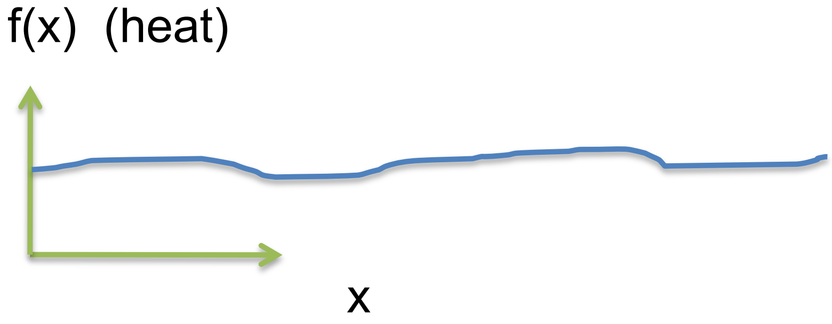


t = 100

Surface characterization

Heat equation

The heat equation: $\frac{\partial f}{\partial t} = -\Delta f$



t = 1000

Surface characterization

Heat equation

The heat equation: $\frac{\partial f}{\partial t} = -\Delta f$

Heat kernel: $K(t,x,y) = \sum e^{-\lambda_i t} \phi_i(x)\, \phi_i(y)$

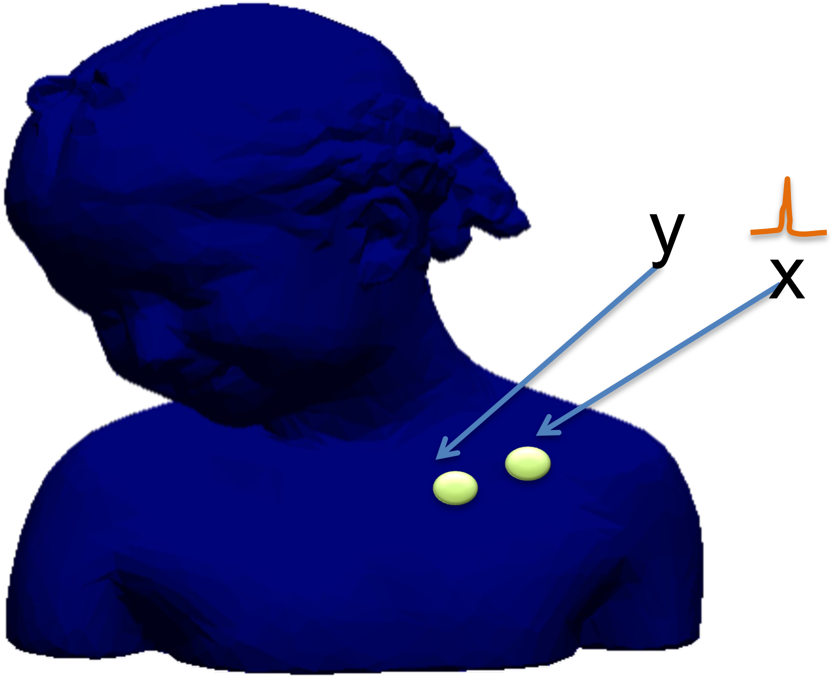
Solution of the heat equation: $f(t,x) = \int K(t,x,y)\, f(0,y)\, dy$

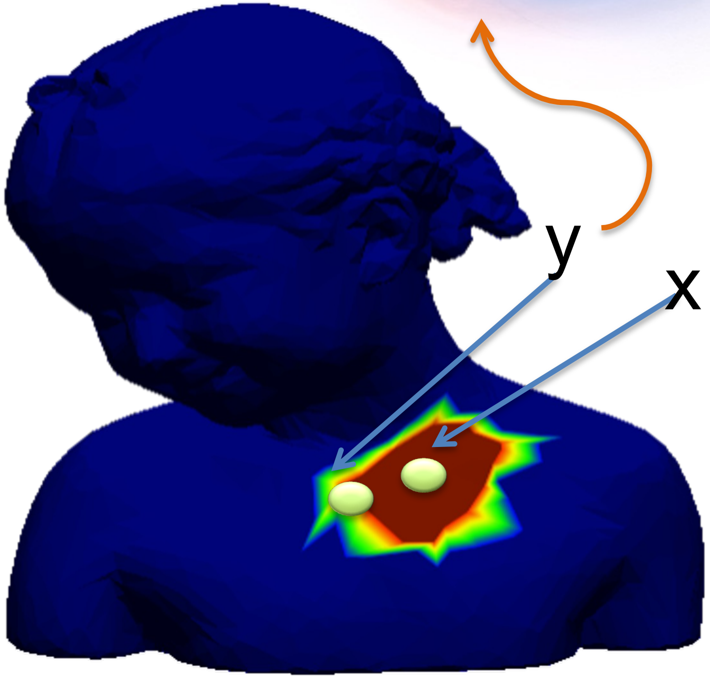
Surface characterization

Heat equation - What is the meaning of K(t,x,y) ?

Initial time: we inject a Dirac of heat at x

How much heat do we get at y after t seconds ? K(t,x,y)





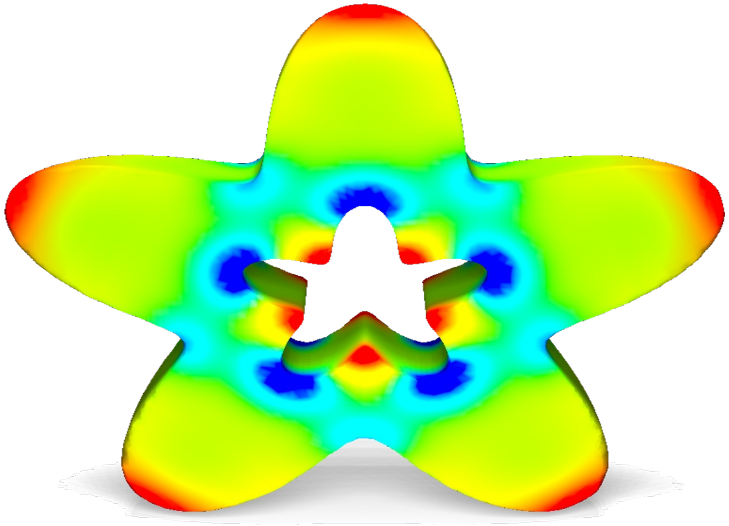
Surface characterization

Heat equation

Heat Kernel Signature [Sun, Ovsjanikov and Gubas 09]
Auto-diffusion [Gebal, Baerentzen, Anaes and Larsen 09]

How much heat remains at x after t seconds ?

$$ADF(t,x) = HKS(t,x) = K(t,x,x) = \sum e^{-\lambda it} \phi_i^2(x)$$



Applications:
- shape signature,
- segmentation using Morse decomposition,
- …

Summary

- Minimizing Rayleigh quotient instead of using « pinning »
  enforces global constraints (moments) that avoid the trivial solution
  - *fieldler vector for streaming meshes [Isenburg et.al]*
  - *Spectral conformal parameterization [Muellen et.al]*

- The notion of fundamental solution plays a … fundamental role.
Strong connections with spectral analysis (and this is what Fourier invented Fourier analysis for !)
  - *Green function / Poisson equation - GPS coordinates [Rustamov]*
  - *Heat kernel signature ,[Sun et.al] / auto-diffusion function [Gebal et.al]*

- More to explore: the Variational Principle (see Wikipedia)