LOCALIZATION OF DISTRIBUTED SOURCES

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Abstract

Most array processing algorithms are based on the assumption that the signals are generated by point sources. This is a mathematical constraint which is not satisfied in many applications. In this paper, we consider situations where the sources are distributed in space with a parametric angular cross-correlation kernel. We propose an algorithm that localizes the distributed sources by estimating their parameter vector. The method is based on minimizing a scalar product between the array manifold and the noise eigenvectors of the correlation matrix. We study two cases corresponding to completely correlated and totally uncorrelated signal distributions. We compare our method to the conventional MUSIC algorithm. The simulation studies show that the new method outperforms the MUSIC algorithm by reducing the estimation bias and the standard deviation.

Résumé

La plupart des algorithmes de traitement d'antenne reposent sur l'hypothèse que les signaux reçus sont générés par des sources ponctuelles. Il s'agit là d'une contrainte mathématique qui n'est pas satisfaite dans plusieurs applications. Dans cet article, nous supposons que les sources sont distribuées spatialement selon un patron de directivité appartant à une classe paramétrique de noyaux de corrélation croisée angulaire. Nous proposons un nouvel algorithme paramétrique pour la localisation des sources distribuées. Cet algorithme est basé sur la minimisation du produit scalaire entre la variété d'antenne et les vecteurs propres du sous-espace bruit. Nous étudions deux cas particuliers correspondants à une distribution spatiale cohérente et non-corrélée, respectivement. Des simulations montrent qui les performances de l'algorithme proposé surpassent considérablement celles de l'algorithme MUSIC.

1. Introduction

In array processing it is frequently assumed that the signals of interest are generated by point sources. This is a modeling assumption that is seldom satisfied in reality and many practical examples can be found where it does not hold. In sonar, the reflection of the signal and the penetration into the lower levels of the seabed create a spatial distribution of the receiving waveform which affects the performance [1]. Another example arises in radar where a short pulse is shot towards the target. When the target is spread in range, the received signal is a superposition of the reflections of the transmitted pulse from different parts of the target, and appears as a distributed source [2].

For narrowband point source configurations, the dimension of the signal subspace, defined as the span of the location matrix, is equal to the number of uncorrelated signals. Thus each source has a one-dimensional representation in the signal subspace. A distributed source can be contemplated as a superposition of a large number of closely-spaced point sources. Thus the corresponding location matrix spans the whole space and the noise subspace is empty. This explains why the conventional array processing techniques such as MUSIC and ESPRIT, which are based on the signal and noise subspace decomposition for point source scenarios, often lead to erroneous results when applied to distributed sources [1].

In this paper we propose a parametric approach to localize distributed narrowband sources using measurements from the array output. We assume that the source distribution pattern is chosen from a parametric class of functions, with each function in this class being uniquely represented by a parameter vector. With this assumption the localization problem becomes the one of parameter estimation. The proposed method is based on minimizing a scalar product between the array manifold and the noise subspace eigenvectors of the correlation matrix. The definition of the array manifold is different from the conventional one used by the MUSIC algorithm and the scalar product is also based on a kernel that introduces an orientation in space. The dimension of the signal subspace is a crucial factor which is required for parameter estimation. We define the concept of time-extension-width product and show that the effective dimension of the signal subspace is directly related to the product of time and extension width.

2. Problem formulation

Consider an array of p sensors monitoring a wave field generated by q spatially distributed narrowband sources in an

additive background noise. For distributed sources the observation vector in the frequency domain is given by

$$\mathbf{x} = \sum_{i=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)s_i(\theta;\psi_i)d\theta + \mathbf{n}, \qquad (1)$$

where $\mathbf{a}(\theta)$ is the $p \times 1$ location vector of the array in the direction θ, $\mathbf{n}$ is the $p \times 1$ additive noise vector, and $s_i(\theta;\psi_i)$ is the angular signal density of the i-th source which is also a function of the angle of arrival, θ, and the parameter vector, ψ_i. Examples of the parameter vectors are the two limits of the directions of arrival (DOA) for a uniform source distribution or the angle of maximum power and -3 dB extension width for a bell-shaped distribution.

For a uniform linear array with the phase reference point at the first sensor, the location vector is given by $\mathbf{a}(\theta) = [1 \; \mu_\theta \; \ldots \; \mu_\theta^{p-1}]^T$, and $\mu_\theta = \exp(j\omega_0 d \sin(\theta)/v)$, where d is the distance between two consecutive sensors, v is the wave speed, ω_0 is the center frequency of the source signal, and T is the transpose operator. If the distance d is equal to half the wavelength at frequency ω_0, μ_θ will be given by $\mu_\theta = \exp(j\pi \sin\theta)$.

The signal and noise samples are modeled as zero-mean, independent, circular, complex Gaussian random variables. The signals and noise are also considered to be uncorrelated from each other. The correlation matrix of the noise is assumed known but for a scalar, σ_n^2. In the sequel, we will consider spatially white noise. Generalization to the nonwhite case is done by using pre-whitening. With these assumptions the correlation matrix of the array output is given by

$$\mathbf{R}_x = \sum_{i=1}^{q}\sum_{j=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)p_{ij}(\theta,\theta';\psi_i,\psi_j)$$
$$\times \mathbf{a}^H(\theta')d\theta d\theta' + \sigma_n^2 \mathbf{I}, \qquad (2)$$

where $\mathbf{I}$ is the $p \times p$ identity matrix and

$$p_{ij}(\theta,\theta';\psi_i,\psi_j) = E[s_i(\theta;\psi_i)s_j^*(\theta';\psi_j)] \qquad (3)$$

is the angular cross-correlation kernel, parameterized by the unknown parameter vectors ψ_i and ψ_j. The superscripts, H, and $*$, represent the Hermitian transpose and the complex conjugation, respectively.

For completely uncorrelated sources, the angular cross-correlation kernel is simplified to

$$p_{ij}(\theta,\theta';\psi_i,\psi_j) = p(\theta,\theta';\psi_i)\delta_{ij}, \qquad (4)$$

where δ_{ij} is the Kronecker delta. In the rest of the paper we assume that the sources are uncorrelated. Note that this is not a restrictive assumption. If the sources are correlated, they can be treated as a single source with a new angular cross-correlation kernel which is the addition of the angular cross-correlation kernel of the sources.

The angular cross-correlation kernel $p_i(\theta,\theta';\psi_i)$ can be simplified for two extreme cases. If the signals arriving from different rays are uncorrelated, the angular cross-correlation kernel can be shown as

$$p(\theta,\theta';\psi_i) = p(\theta;\psi_i)\delta(\theta - \theta'), \qquad (5)$$

where $\delta(\theta)$ is the Dirac delta function. Since this model can be applied to scattering media, we call this the *Scatter*

Distributed (SD) signal. On the other hand, if the signal rays at different angles are coherent, the kernel is separable and it can be written as

$$p(\theta,\theta';\psi) = \sum_{i=1}^{q} \eta_i g_i(\theta;\psi_i)g_i^H(\theta';\psi_i), \qquad (6)$$

where η_i is a scalar value representing the power of the i-th source observed at the reference point of the array, and $g_i(\theta;\psi_i)$ is a complex deterministic angular signal density defined in the interval $[\frac{-\pi}{2} \; \frac{\pi}{2}]$ and normalized according to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\theta;\psi_i)d\theta = 1. \qquad (7)$$

For a point source scenario, the angular signal density $g(\theta;\psi)$, is replaced by $\delta(\theta - \theta_0)$. A signal with the angular correlation kernel (6), is called the *Coherently Distributed* (CD) signal.

In the following section we propose a localization technique for CD and SD signals. Note that in practice an intermediate situation might occur that corresponds to a partially correlated signal where the rays of signal which are arriving from different angles are partially correlated. We will show that the partially correlated signal can also be detected and localized using the same method as the SD signal.

3. Localization

Assume that the observation space can be decomposed into the signal and the noise subspaces and the noise eigenvectors of $\mathbf{R}_x$ are shown by $\mathbf{E}_n$. For distributed sources we propose the following criterion. The parameter vector is found from

$$\psi = \arg\max_{\psi} \frac{1}{\|\mathbf{a}^H(.)\mathbf{E}_n\|_k^2} \qquad (8)$$

where

$$\|s\|_k^2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s(\theta)p(\theta,\theta';\psi)s^*(\theta')d\theta d\theta'. \qquad (9)$$

It is seen that this criterion is similar in form to the spatial spectrum of the MUSIC algorithm with the difference that it uses a weighted norm. In an expanded form the criterion is represented by

$$\psi = \arg\max_{\psi} \frac{1}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n p(\theta,\theta';\psi)\mathbf{E}_n^H \mathbf{a}(\theta')d\theta d\theta'}. \qquad (10)$$

We call this method the Distributed Signal Parameter Estimator (DSPE).

For the CD sources, the rays of arriving waves at different angles are delayed and amplified version of the same signal. In such a case the signal subspace is spanned by the eigenvectors of the correlation matrix corresponding to the q largest eigenvalues. The localization criterion (10) for the CD sources with the angular signal density $g(\theta;\psi)$, is represented as

$$\psi = \arg\max_{\psi} \frac{1}{\mathbf{h}^H(\psi)\mathbf{E}_n \mathbf{E}_n^H \mathbf{h}(\psi)} \qquad (11)$$

where

$$\mathbf{h}(\psi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)g(\theta;\psi)d\theta. \qquad (12)$$

The criterion (11) can be simply written as

$$\hat{\psi} = \arg\max_{\psi} \frac{1}{\|\mathbf{h}^H(\psi)\mathbf{E}_n\|^2} \qquad (13)$$

which is similar in form to the spatial spectrum of the MU-SIC algorithm. The difference is that the array manifold vector for the distributed source is the integral of the product of the location vector, $\mathbf{a}(\theta)$, and the angular signal density.

For the SD sources, the noise subspace is null and the algorithms which are based on the signal and noise subspace decompositions cannot be applied. However, for the special case of uniform distribution we show that with proper choice of the signal and noise subspaces it is possible to use the DSPE algorithm. In the sequel, we will define an effective dimension for the signal subspace. Although the whole observation space is occupied by the signal components, it is possible to show that most of the energy of the signal is concentrated in a few eigenvalues. The number of these eigenvalues is denoted as the effective dimension of the signal subspace which can be used in the localization algorithm.

We present a lemma to find the cross-correlation function at a linear continuous array. For a continuous array, the signal is observed at all the points in the interval $[-\frac{L}{2}\ \frac{L}{2}]$ where L is the entire array length.

Lemma 1: *For an SD source uniformly distributed in the interval $[\theta_0 - \Delta, \theta_0 + \Delta]$, and a noise-free environment, the spatial cross-correlation function at the two observation points, z and z', is given by*

$$E[x(z)x(z')] = \frac{1}{\Delta} e^{j\frac{2\pi}{\lambda}(z-z')\sin\theta_0} \text{sinc}[\frac{2}{\lambda}(z-z')\Delta\cos\theta_0].$$
$$(14)$$

To find the effective dimension of the signal subspace we need the eigenvalue analysis of (14) which can be performed by solving

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{\Delta} e^{j\frac{2\pi}{\lambda}(z-z')\sin\theta_0} \text{sinc}[\frac{2}{\lambda}(z-z')\Delta\cos\theta_0]$$
$$\phi_n(z')dz' = \mu_n\phi_n(z). \qquad (15)$$

The eigenvalues of (15) are the *radial prolate spheroidal functions* [3]

$$\mu_n = \frac{L}{\Delta}[R_{0n}^{(1)}(c,1)]^2 \qquad (16)$$

where c is a parameter defined by

$$c = \pi\Delta\frac{L}{\lambda}\cos\theta_0. \qquad (17)$$

For a uniform linear array with half the wavelength spacing between sensors, (17) can be written as

$$c = \frac{\pi}{2}\Delta(p-1)\cos\theta_0. \qquad (18)$$

In many practical applications the extension width is usually small (less than a few degrees). For $2\Delta < \pi/30$, the parameter vector c is bounded by

$$c < \frac{\pi^2}{120}p. \qquad (19)$$

For a fixed c the radial prolate spheroidal function decreases exponentially with n. From tables of the prolate spheroidal functions [4] it can be seen that for $c \le 4$, about
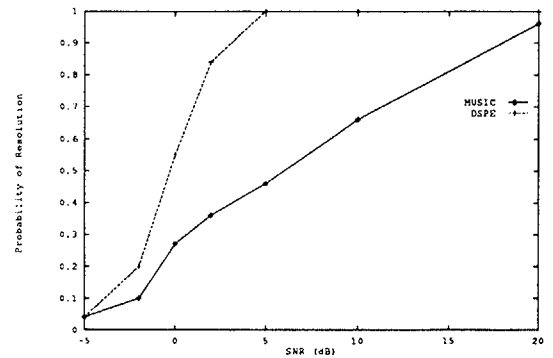

Fig. 1 The probability of resolution for the conventional MUSIC and the DSPE versus SNR.

95 percent of the energy is concentrated in the first $\lceil c \rceil$ eigenvalues where $\lceil c \rceil$ indicates the smallest integer number larger than c. Thus it is seen that the effective number of eigenvalues is bounded and the dimension of the signal subspace depends on the parameter, c. From (18), the parameter c is proportional to $\Delta(p-1)$ which is proportional to the product of the extension width and the observation time across the array.

Once the signal subspace dimension is determined, the DSPE algorithm can be used to localize the sources. It should be noted that in practice the exact value of $\Delta(p-1)$ is not required. If the maximum extension width is known, the upper bound of c can be used to estimate the subspace dimensionality. An easier way is to simply choose the prominent eigenvalues of the correlation matrix regardless of the number of signals.

For partially correlated distributed signals, the signal subspace dimension is between 1 and $\lceil c \rceil$. With choosing an upper bound for the signal subspace dimensionality, the DSPE algorithm still can be used for the partially correlated distribution.

4. Simulation results

We investigate a configuration with two equipower uncorrelated distributed narrowband sources arriving at a linear array of 20 sensors. The spacing between adjacent sensors is equal to half the wavelength at the operating frequency. It is assumed that the sources are coherently distributed in space with the angular correlation Butterworth kernel

$$p(\theta; \psi_i) = \frac{K_i^2}{1 + (\frac{\theta-\theta_i}{\Delta_i})^2} \qquad (20)$$

which is a practical choice [5]. In this equation K_i is a normalization factor, θ_i is the central angle of arrival, and Δ_i is the 3-dB extension width for the i-th source. The central angles of arrival for the two significantly overlapped sources are 10 and 13 degrees with the 3-dB extension widths 1 and 2 degrees, respectively.

A Monte-Carlo simulation of 50 independent runs with 50 collected snapshots for each trial was performed for different signal-to-noise ratios. The resolution performance of the conventional MUSIC and the DSPE are compared in Fig. 1. For high probability of resolution the resolution threshold for
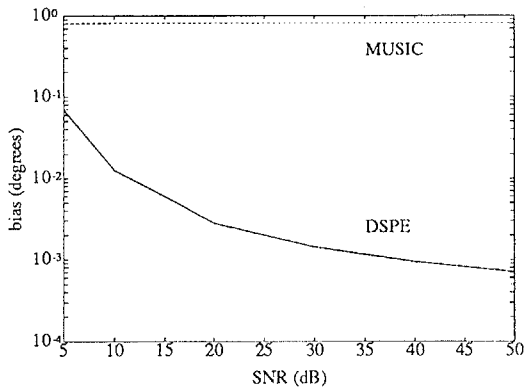
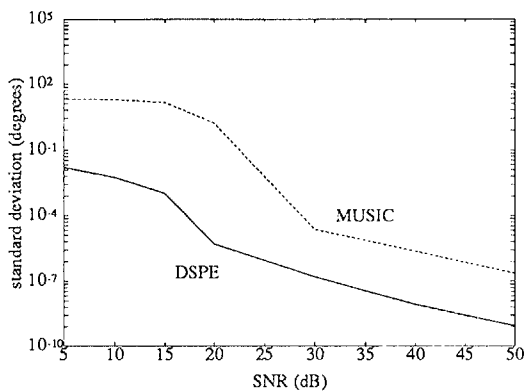**Fig. 2** The bias of estimation versus SNR for the source at 10 degrees.

Fig. 3 The standard deviation versus SNR for the source at 10 degrees.

Fig. 4 The eigenvalues of a configuration with two SD signals.
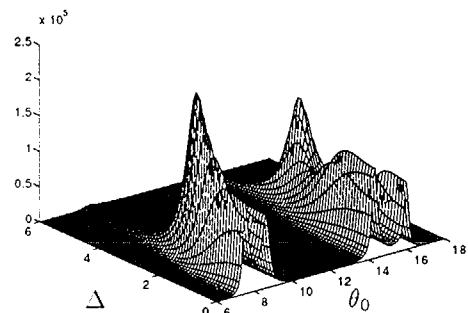
Fig. 5 The spectrum of the DSPE algorithm. The angles are in degrees.

DSPE is about 15 dB lower than the conventional MUSIC algorithm.

For the same configuration we found the bias and the standard deviation of the MUSIC and the DSPE algorithms. The estimation bias for the central DOA of the source at 10 degrees is shown in Fig. 2. A similar result has been found for the source at 13 degrees. It is seen that the estimated DOA is biased in the conventional MUSIC algorithm. The bias cannot be decreased appreciably by increasing the SNR. The DSPE algorithm provides an almost unbiased estimate of the DOA. Moreover, the bias can be reduced by increasing the SNR. The standard deviation of the estimators is compared in Fig. 3. It is seen that the DSPE algorithm is more robust.

For the SD signal scenario we examined a configuration with two uniformly distributed sources arriving at an array of 20 sensors from 8 and 15 degrees with the extension widths 2 and 3 degrees, respectively. The signal-to-noise ratio is 30 dB and 200 snapshots are observed. From (19) the parameter c is smaller than 1.6 for a single source with 3 degree extension. Thus the effective dimension of the signal subspace is equal to 4. The eigenvalues of the sample correlation matrix for this scenario are shown in Fig. 4. It is seen that the first 4 eigenvalues are prominent. The PSDE algorithm is run for this example with 16 noise eigenvectors. The spectrum is illustrated in Fig. 5. The two distinct peaks estimate the location of the signals at 7.92 and 15.04 degrees with the extension widths of 1.86 and 3.10 degrees,
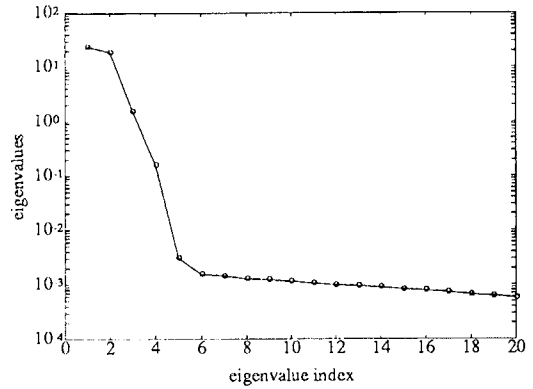
respectively.

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