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Estimation of the Probability Density Function by Spectral Analysis: A Comparative Study

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RÉSUMÉ

Une méthode plus simple est proposée pour mesurer la densité de probabilité **pdf** d'un processus aléatoire. La méthode fait usage de la analyse du spectre énergétique et d'une théorème sur la modulation de fréquence, lequel est presenté avec une rigoureuse formulation. La consistence de la technique spectralle, aussi que sa suffissance, est analysée dans cette article. Des resultats pratiques sont presentés pour permettre la comparaison avec l'approache théorique.

ABSTRACT

A simple method is proposed to measure the probability density function (**pdf**) of random signals. The method uses spectral analysis and a theorem on fequency modulation, which is presented with a novel and more rigorous formulation. The consistency of the spectral technique, as well as its sufficiency, is analysed in this article. Some practical results are presented to compare the theoretical approach.

1 Introduction

The time series approach is commonly used to access the estimation of the probability density function [1]. Many methods have been proposed as useful estimates for the **pdf**. Typical procedures are the kernel method, the orthogonal series method, and the interpolation method [2]. This is usually done by measurement of the time spent by the signal between two specified levels or through a pulse counting process, for discrete signals. This leads to biased and inconsistent estimates, and to mean square errors that depend on the **pdf** itself [3]. It is a common practice to assume the stationarity and ergodicity of the random process into analysis [4].

This paper is concerned with the presentation of a new method of measuring the probability density function of random signals, and establishing a new bound on the estimation error, using a theorem on frequency modulation, correlation techniques and spectral analysis [5] [6]. The proposed method is based on the spectral analysis of the random process. The estimation error upper bound is shown to decrease steadly as the modulating index is increased, which implies a decrease in frequency or an increase in the power of the signal [7].

Some results are presented , comparing the computacional simulation of the method with the experimental setup. The simulation performed in FORTRAN, on a workstation [8]. The experimental setup used a Philips radio as modulator, with $70MHz$ of intermediary frequency, a noise generator and a waveform generator. The experimental results were obtained from HP 8553B spectrum analyser.

2 Procedure for Estimating the Probability Density Function

A well known theorem asserts that the spectrum of a high-index frequency modulated waveform can be approximated by the probability distribution of its instantaneous frequency deviation [6]. The method developed in this section is proposed to estimate and measure the first order probability density function (**pdf**) of random signals which utilizes the previous theorem. A new proof is presented which includes the linear mean square estimator, and a new upper bound on the estimation error based on Papoulis' inequality [9].

In the proposed model, the modulated signal $s(t)$ is obtained from the following equations:

$$s(t) = A\cos(w_c t + v(t) + \phi) \tag{1}$$

$$v(t) = D.\int_{-\infty}^{t} m(t)dt \tag{2}$$

where the constant parameters A, w_c and D represent respectively the carrier amplitude, frequency (rd/s) and frequency deviation index. The signal whose **pdf** one intends to analyse is represented by $m(t)$, here considered a zero mean random stationary process. The phase of the carrier ϕ is random, uniformly distributed in the range $(0, 2\pi)$ and statistically independent of $m(t)$.

The following steps are observed in the evaluation of the probability density function (**pdf**) of $m(t)$:

1. Compute the autocorrelation function of equation 1.
2. Obtain an estimate of this autocorrelation, for the case of a high modulation index, using the linear mean square estimator [9].
3. Compute the power spectrum density (**PSD**) of $s(t)$ by application of the Wiener-Khintchine theorem on the autocorrelation estimate [10].

The power spectrum density of $s(t)$ is demonstrated to approach the probability density function of $m(t)$ in equation 2, as the modulation index is increased. The difference between the estimate of the autocorrelation function in step 2, and the actual autocorrelation in step 1 is the estimation error E_S. An upper bound for this error is evaluated in the next section and is shown to decrease with the an increase in the modulation index β.

A high modulating index causes a spectrum broadening of the modulated signal. In addition, the modulated carrier **PSD** turns into the probability density function of the modulating signal. That is the main result of Woodward's theorem, and will be discussed in the following. The autocorrelation function of $s(t)$, defined by equation 1 is shown below

$$R_S(\tau) = E[s(t)s(t+\tau)] \tag{3}$$

Evaluation of the autocorrelation in terms of $v(t)$ leads to

$$\begin{aligned} R_S(\tau) &= \frac{A^2}{4} e^{jw_c\tau} E[e^{j(-v(t)+v(t+\tau))}] \\ &+ \frac{A^2}{4} e^{-jw_c\tau} E[e^{j(v(t)-v(t+\tau))}] \end{aligned} \tag{4}$$

The linear mean square estimator was defined in [9] as

$$v(t+\tau) \approx \frac{R_V(\tau)}{R_V(0)} v(t) - \frac{R'_V(\tau)}{R''_V(0)} v'(t) \tag{5}$$

After some algebraic manipulation and simplification, one obtains the following expression, where $0(.)$ stands for the remaining terms of the series

$$v(t+\tau) - v(t) \approx \tau v'(t) + O(\tau^2) \tag{6}$$

Use of the linear mean square estimator in equation 4 gives

$$\hat{R}_S(\tau) = \frac{A^2}{4}e^{jw_c\tau}E[e^{j\tau v'(t)}] + \frac{A^2}{4}e^{-jw_c\tau}E[e^{-j\tau v'(t)}]. \quad (7)$$

But, $v'(t) = dv(t)/dt = z$, where z is the carrier frequency deviation, thus

$$\hat{R}_S(\tau) = \frac{A^2}{4}e^{jw_c\tau}E[e^{j\tau z}] + \frac{A^2}{4}e^{-jw_c\tau}E[e^{-j\tau z}]. \quad (8)$$

Recalling the expression for the expectancy,

$$E[e^{j\tau z}] = \int_{-\infty}^{\infty} p_Z(z)e^{j\tau z}dz \quad (9)$$

where $p_Z(z)$ is the probability density function of process $z = dv(t)/dt$ One can therefore evaluate equation 8 and obtain

$$\hat{S}_S(w) = \frac{A^2}{2}[p_Z(w + w_c) + p_Z(w - w_c)]. \quad (10)$$

Expression 10 allows estimating the **pdf** of the modulating signal derivative.

The definition of z, as stated above, yields

$$p_Z(z) = \frac{1}{D}p_M(\frac{m}{D}) \quad (11)$$

where $p_M(.)$ stands for the **pdf** of $m(t)$.

Substituting equation 11 into equation 10 gives

$$\hat{S}_S(w) = \frac{A^2}{2D}[p_M(\frac{w + w_c}{D}) + p_M(\frac{w - w_c}{D})]. \quad (12)$$

A Gaussian **pdf** for $m(t)$ leads to

$$p_M(m) = \frac{1}{\sqrt{2\pi P_M}}e^{(-\frac{m}{2P_M})^2} \quad (13)$$

with $P_M = R_M(0)$ = power of signal $m(t)$.

The equation for a carrier frequency-modulated by a Gaussian process is given below, which is the very **pdf** of signal $m(t)$.

$$\hat{S}_S(w) = \frac{A^2}{2D\sqrt{2\pi P_M}}e^{-\frac{(w \pm w_c)^2}{2D^2P_M}}. \quad (14)$$

A plot os the Gaussian signal is depicted in Figure 1. The computer output for the **pdf** is illustrated in Figure 2. From equation 14 the RMS frequency deviation is given by $D\sqrt{P_M}$. As a matter of comparison, the actual spectrum is shown in figure 3.

For the sinusoidal modulating signal shown in Figure 4

$$p_M(m) = \frac{1}{\pi\sqrt{V^2 - m^2}}, |m| < V. \quad (15)$$

That leads to the following **PSD**

$$\hat{S}_S(w) = \frac{1}{2\sqrt{V^2D^2 - (w \pm w_c)^2}}, |w - w_c| < D.V. \quad (16)$$

Expression 16 above, whose simulation is shown in Figure 5, is in good agreement the with experimental results of Figure 6.

3 A new upper bound on the estimation error

In the preceding section an expression has been derived for the **PSD** of a carrier modulated in frequency by a signal $m(t)$. For a high modulating index it was observed that the **PSD** approximates the **pdf** of the modulating signal. This section is concerned with estimating the approximation error, in order to validate the procedure.

First, one can show that the mean square estimator used in equation 7 is efficient, consistent and unbiased. It is sufficient to demonstrate that

$$E[v(t+\tau) - \frac{R_V(\tau)}{R_V(0)}v(t) - \frac{R_V'(\tau)}{R_V''(0)}v'(t)] = 0 \quad (17)$$

$$E[(v(t+\tau) - \frac{R_V(\tau)}{R_V(0)}v(t) - \frac{R_V'(\tau)}{R_V''(0)}v'(t))^2] \to 0. \quad (18)$$

Expression 17 assures that the expected value of the estimator equals the mean of the signal. Its trivial to prove this assumption. The second expression yields a minimum mean square error, and can be shown to be

$$e^2 \leq (w_M\tau)^2R_V(0) + \frac{(R_V'(\tau))^2}{R_V''(0)}. \quad (19)$$

or

$$e^2 \leq (w_M\tau)^2R_V(0) \quad (20)$$

The last result indicates that the error for the mean square estimator goes to zero as $\tau \to 0$, but depends on the square of the signal bandwidth.

The next step is to show how the approximation behaves as the modulation index increases. Utilization of a linear mean square estimator in equation 4 yields an approximation error given by

$$E_S(\beta) = S_S(w) - \hat{S}_S(w) \quad (21)$$

Considering the limiting case ($\tau = \pi/\beta w_M = 1/\beta f_M$), an upper bound on the normalized error can be determined. Substituting the expressions for $S_S(w)$ and $\hat{S}_S(w)$ obtained from equations 4 and 7 into equation 21, evaluating the expectancies at w_c = 0and using the following inequality in equation 21 [9]

$$\tau^2E[(v'(t))^2] \geq E[(v(t+\tau) - v(t))^2] \geq \frac{4\tau^2}{2\pi^2}E[(v'(t))^2] \quad (22)$$

leads to

$$E_S(\beta) \leq [\frac{\pi - 2}{\beta w_M}][1 - 2Q(\frac{\pi}{\beta w_M})] \quad (23)$$

where $Q(x)$ is the Q-function, defined by

$$Q(x) = \frac{1}{2\pi}\int_x^{\infty} e^{\frac{x^2}{2}}dx \quad (24)$$

The above expression is a very tight bound and shows the error dependency on the modulating index β. In fact, the spectrum of the modulating signal is lower and upper bounded by Gaussian curves. It is interesting to compare a plot of equation 23 with a previous upper bound on the error, under the same conditions [5].

4 Conclusions

A method for estimating and measuring the probability density function (**pdf**) of random signals, with application to communications, has been derived. The method is based on a new presentation of a known theorem and has been tested in practice with good results [11]. A digital computer implementation of the method was performed through contract No. C.NE.085.16 with EMBRATEL.

The efficiency of the estimation used is assured because the variance goes to zero. The estimation always gets better as β goes to infinity, which implies the consistency of the method. All the relevant information is available for the estimation, giving sufficiency to the estimator. A new upper bound was introduced on the estimation of the probability density function through spectral analysis. Some simulation results were presented to illustrate the procedure, and a comparisson was made with results from previous work.

On the other hand, the modulated signal bandwidth increases as the modulation index is increased. This implies in an incressing number of points needed to represent accurately the pdf of the signal.

The linear increment in the bandwidth is well compensated by the exponential decrease in the approximation error with the modulation index. Considering that the number of points used in representing the spectrum is half the required number of time points, a reduction in the aproximation error implies an increase in the observed number of points for the signal. A very fast algorithm can be devised for estimating the probability density function of signals by using techniques of FFT in conjunction with the proposed procedure.

As established in this paper, the power spectrum density (PSD) of the frequency-modulated waveform approximates the pdf of the modulating signal when the modulating index increases, and reaches an error smaller than 0.08% for an index of 10. In the experimental setup use is made of a Philips radio system as modulator, with $70MHz$ of intermediary frequency (IF), a noise generator in the range of $316kHz$ to $8,204kHz$ and a waveform genarator. The figures presented in the work were obtained from photos taken from the Hewlett Packard spectrum analyser 8553B.

5 Aknowledgment

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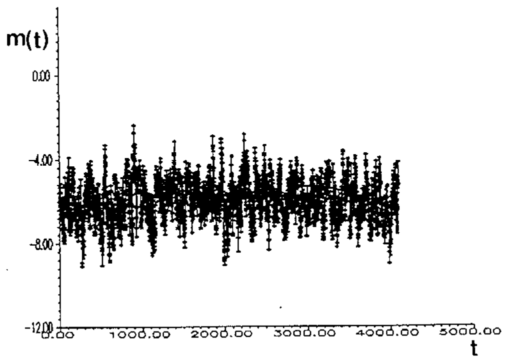

Figure 1: Gaussian signal, obtained by simulation

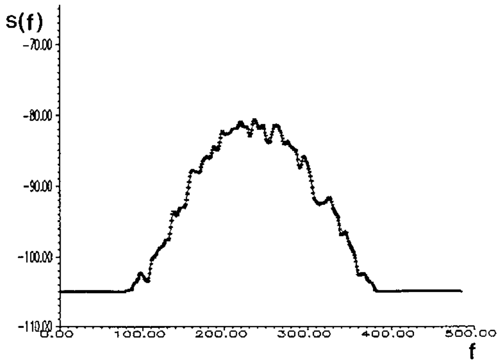

Figure 2: Probability density function of a Gaussian signal, obtained by simulation

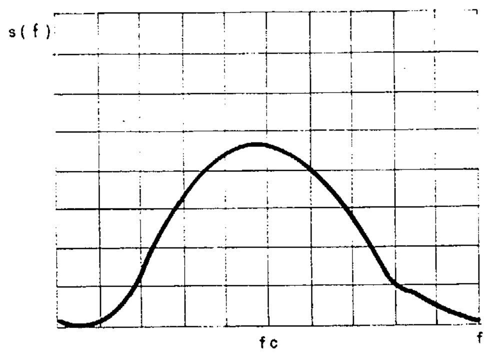

Figure 3: Probability density function of a Gaussian signal, obtained during the experiment

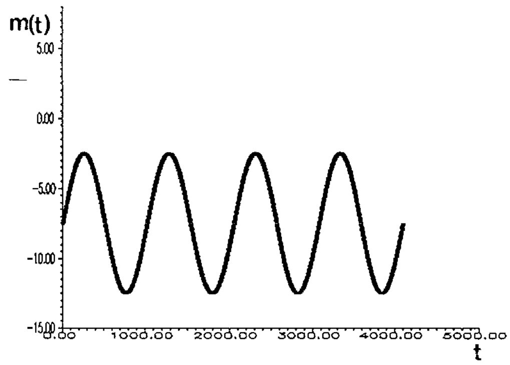

Figure 4: Sinusoidal signal, obtained by simulation

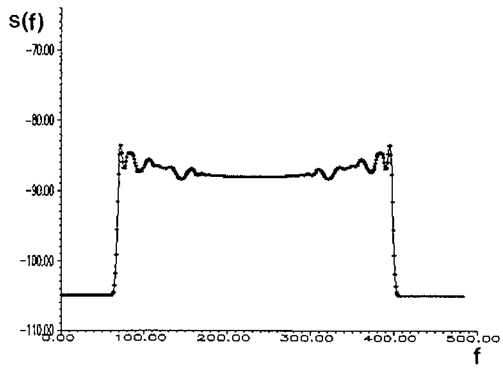

Figure 5: Probability density function of a sinusoidal signal, obtained by simulation

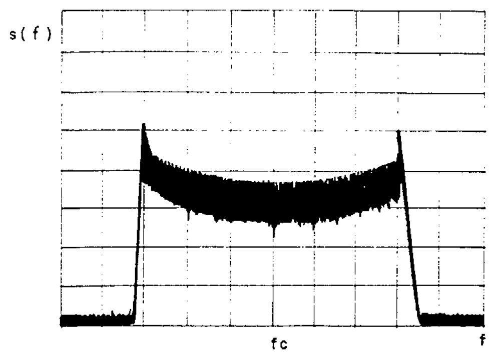

Figure 6: Probability density function of a sinusoidal signal, obtained using the spectrum analyser

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