# OPTIMUM AND SUBOPTIMUM DETECTION OF WEAK CYCLOSTATIONARY SIGNALS IN NON-GAUSSIAN NOISE

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RESUME

On s'intéresse ici à l'étude des algorithmes localement optimaux de détection d'un signal cyclostationnaire noyé dans un bruit non gaussien. Dans ce cas le détecteur localement optimal demande une sommation de termes complexes dont les phases dépendent de la phase de chacune des périodicités du signal à détecter.

Dans le but de simplifier l'implémentation de la structure envisagée, on synthétise un détecteur sous-optimal ("single-cycle detector") qui considère seulement une périodicité du signal à détecter et, par conséquent, èvite l'estimation des phases.

La performance du détecteur sous-optimal proposé est evaluée. De plus, dans le but de le comparer avec le détecteur localement optimal synthétisé en se référant à un signal stationnaire et avec le détecteur quadratique, des expressions pour le coefficient d'efficacité asymptotique sont obtenues.

SUMMARY

The paper deals with the locally optimum detection of cyclostationary signals operating in non-Gaussian noise. The locally optimum detector for such kind of signals requires a summation of complex quantities the phases of which depend on the phase of each of the periodicities in the signal to be detected. Since its physical implementation could be quite involved, it is synthesized a suboptimum single-cycle detector that,by considering only one periodicity of the signal to be detected, avoids the phase estimates.

The performance of the proposed single-cycle detector is assessed. Moreover, in order to make a comparison with the locally optimum detector synthesized with reference to a stationary signal and with the square-law detector, expressions of the asymptotic relative efficiency are derived.

## 1. INTRODUCTION

In the last few years the signal reception in non-Gaussian noise has received great attention, in that there are many situations for which the Gaussian noise model is not adequate |1-3|. In such cases the weak-signal assumption has allowed to derive suboptimum detection structures referred to as locally optimum (LO) |4-8| and asymptotically optimum (AO) |8-10| detectors, that are particularly interesting for their easier physical implementation and for their characteristics of asymptotical optimality. More precisely, the synthesis and the performance analysis of LO and AO detectors have been extensively carried out on the assumption of signal to be detected modeled as known signal, for both lowpass |5,11| and bandpass (coherent and noncoherent) signals |6,7,10,11|. It has been shown that the derived structures are slight modifications of the conventional detectors, i.e., those optimized against Gaussian noise. In fact, in such structures the conventional receiver (e.g.,matched filter, square-law detector, etc.) follows a processing of the received signal performed by means of a zero-memory non-linearity (ZMNL) whose characteristic depends only on the noise statistics.

In many cases of interest, as, for example, the detection (interception) of weak spread-spectrum signals, owing to the low signal-to-noise ratio (SNR) of the received signal, long observation times are required for reliable detection performance. Therefore, since the particular message embedded in a long segment of a signal is not known in advance and, hence, the finite but immense number of sequences does not allow to assume the signal as known, it is suitable to model it as a random process.In such cases,as it is well-known, radiometric techniques have commonly utilized to detect the signal presence by employing an increase, above the ambient noise level, of the received energy in certain spectral bands.

On the assumption of stationarity of the signal to be detected, the radiometric detector (which reduces to the square-law (SL) detector when the signal is assumed white) is a LO detector in a Gaussian noise environment. Such an assumption, encountered often in the literature (see |11| and the papers there referenced) also for the signal detection in non-Gaussian noise, leads to a structure which differs from the conventional radiometer for the presence of a ZMNL pre-processing of the received signal.

The stationary signal model, however, does not account for the signal periodicities which arise from usual signal processing operations(e.g.,sampling, scanning, modulating, coding, etc.).Therefore, recently, in order to exploit the periodicities,a cyclostationary or almost cyclostationary model has been assumed for the signal to be detected. A LO detector, with reference, however, to the Gaussian noise, has been considered and its quadratic detection statistic has been re-expressed in terms of cyclic autocorrelation functions of the signal to be detected |12|. This has allowed to recognize that the LO detector is a multi-cycle (MC) detector whose corresponding test statistic uses all the possible spectral lines that can be generated from the received data by means of a quadratic transformation. A suboptimum detector of easier implementation, referred to as single-cycle (SC) detector, has been also proposed |12|. It can outperform |13| the radiometer (i.e., the LO structure synthesized under the assumption of stationary signal and Gaussian noise) when they operate in an environment characterized by unknown and changing noise level and interference activity.

In this paper, with reference to the LO detection of cyclostationary signals, the Gaussian noise assumption considered in |12,13| is removed. The LO detection statistic, which is a quadratic form, is re-expressed (Section 2) in terms of cyclic autocorrelation functions of the signal to be detected.

The LO detector for cyclostationary signals is a multi-cycle detector which requires a summation of complex quantities the phases of which depend on the phase of each periodicity present in the signal to be detected. Since its physical implementation could be quite involved, in Section 2 a suboptimum single-sycle detector is also proposed. It avoids the phase estimates by considering only one periodicity of the signal to be detected.

The performance of the proposed SC detector is assessed (Section 3) in terms of deflection that is a useful measure of the output SNR in weak-signal situations. Moreover, in order to make a comparison between the performance of the SC detector and that of the LO receiver synthesized under the assumption of stationary signals in non-Gaussian noise, an expression of the asymptotic relative efficiency (ARE) |4| is provided. Finally, since the SL detector is often utilized in the applications, a comparison is made (again in terms of ARE) between such a detector and the SC one when they operate in a non-Gaussian noise environment.

## 2. MULTI-CYCLE AND SINGLE-CYCLE DETECTORS

The detection problem under consideration can be represented by the following hypothesis test:

$$H_0 : y_i = n_i \qquad i = 1,2,\dots,M \qquad (1)$$
$$H_1 : y_i = s_i + n_i$$

where $y_i$ and $n_i$ denote the ith sample of the received signal and the noise, respectively. The random variables (rv's) $n_i$ are assumed to be independent and identically distributed (iid). Finally, $s_i$ denotes the ith sample of the signal to be detected, which is modeled as zero-mean cyclostationary or almost cyclostationary process.

By assuming that $n_i$ and $s_i$ are statistically independent, on the previous assumptions, the sufficient statistic of the LO detector |4| is given by:

$$\sum_{i,m=1}^{M} [g(y_i)\, g(y_m) + g'(y_i)\delta_{im}]K_s(i,m) \qquad (2)$$

where $\delta_{im}$ is the Kronecker delta and

$$g(y) \underline{\Delta}\, f'(y)/f(y) \qquad (3)$$

with $f(\cdot)$ denoting the probability density function (pdf) of the rv $n_i$. Moreover, in (2) $g'(\cdot)$ and $f'(\cdot)$ are the derivative of $g(\cdot)$ and $f(\cdot)$, respectively, and $K_s(i,m)$ is the i-m$^{th}$ element of the autocorrelation matrix of the signal, i.e.,

$$K_s(i,m) \underline{\Delta}\, E[s_i s_m] \qquad (4)$$

where $E[\cdot]$ denotes the statistical expectation.

The almost cyclostationarity assumption allows us to express $K_s(i,m)$ in the Fourier-series form

$$K_s(i,m) = \sum_{\alpha} K_s^{\alpha}(i-m) \exp[j\pi\alpha(i+m)] \qquad (5)$$

where

$$K_s^{\alpha}(i) \underline{\Delta} \lim_{N\to\infty} (2N+1)^{-1} \sum_{r=-N}^{N} K_s(i+r,r)\exp[-j2\pi\alpha(r+\frac{i}{2})] \qquad (6)$$

is the cyclic autocorrelation function |14| and the index $\alpha$ of the summation ranges over all the harmonics (integer multiples) of the fundamental frequencies of cyclostationarity such as carrier frequencies, chip rates, code repetition rates, etc.

By substituting (5) in (2), one has the following expression for the LO detector statistic:

$$\sum_{\alpha} \sum_{i,m=1}^{M} K_s^{\alpha}(i-m)[g(y_i)\, g(y_m) + g'(y_i)\delta_{im}] \exp[j\pi\alpha(i+m)]$$

$$= \sum_{\alpha} M \sum_{r=1-M}^{M-1} K_s^{\alpha}(r)^* R_g^{\alpha}(r) + M \sum_{\alpha} K_s^{\alpha}(0)^* A_{g'}^{\alpha} \qquad (7)$$

where the asterisk identifies the complex conjugate, $R_g^{\alpha}(\cdot)$ is the cyclic correlogram of the signal at the output of the ZMNL

$$R_g^{\alpha}(r) \underline{\Delta}\, M^{-1} \sum_{i=1}^{M-|r|} g(y_i)g(y_{i+|r|}) \exp[-j2\pi\alpha(i+|r/2|)] \qquad (8)$$

and, finally,

$$A_{g'}^{\alpha} \underline{\Delta}\, M^{-1} \sum_{m=1}^{M} g'(y_m)\exp(-j2\pi\alpha m) \qquad (9)$$

is an estimate of the sinewave component with frequency $\alpha$ of the signal $g'(y_m)$.

Equation (7) shows that the LO sufficient statistic in non-Gaussian noise consists of a summation of two terms. In the former one evaluates, for all cycle frequencies $\alpha$ of the signal to be detected (MC detector), the cyclic correlograms (8) of the signal at the output of the ZMNL $g(\cdot)$ and, then, performs the summation (over $\alpha$) of the correlations between each cyclic correlogram and the corresponding cyclic autocorrelation function $K_s^{\alpha}(\cdot)$. In the latter one performs a weighted summation (over $\alpha$) of the estimates $A_{g'}^{\alpha}$, the weights of which are $K_s^{\alpha}(0)^*$.

Let us note that, for stationary signals, (7) is in agreement with previous results |11|, in that in such a case $K_s^{\alpha}(\cdot)=0$ for any $\alpha \neq 0$ and $K_s^{0}(\cdot)$ is the autocorrelation function $K_s(\cdot)$.

In (7) the phases of the complex quantities $K_s^{\alpha}(\cdot)$ for $\alpha \neq 0$ depend on the phases of the signal periodicities. Therefore, the implementation of the MC detector can be quite involved in that it requires phase estimates of the periodicities of the signal to be detected. In order to avoid such estimates, it is suitable to consider a suboptimum detection structure, which employes only one periodicity of the signal (i.e., only one value of $\alpha (\neq 0)$ and its opposite) and,therefore, it is referred to as single-cycle detector |12,13|. In such a case by using

$$K_s^{-\alpha}(r) = K_s^{\alpha}(r)^* \qquad (10)$$

from (7) it follows that

$$Re(Z) \underline{\Delta}\, Re\Big\{\sum_{i,m=1}^{M} K_s^{\alpha}(i-m)[g(y_i)g(y_m) + g'(y_i)\delta_{im}]e^{j\pi\alpha(i+m)}\Big\}$$

$$= M\, Re\Big\{\sum_{r=1-M}^{M-1} K_s^{\alpha}(r)^* R_g^{\alpha}(r) + K_s^{\alpha}(0)^* A_{g'}^{\alpha}\Big\} \qquad (11)$$

where $Re(\cdot)$ denotes the real part.

As previously noted, the SC detector statistic (11) does not require the knowledge of the time origin of the signal to be detected, as it follows by considering that, if one assumes $w(t) \underline{\Delta}\, s(t-t_0)$ with an arbitrary choice of $t_0$, one has

$$K_w^{\alpha}(\tau) = K_s^{\alpha}(\tau)\exp(-j2\pi\alpha t_0) \qquad (12)$$

With reference to a cyclostationary signal embedded in Gaussian noise, since the ZMNL is given by

$$g(y) = -y/\sigma_n^2 \qquad (13)$$

where $\sigma_n^2$ is the variance of each noise sample, from (11) it results that the SC detector statistic is

$$Re\Big\{\sum_{r=1-M}^{M-1} K_s^{\alpha}(r)^* R_y^{\alpha}(r)\Big\} \qquad (14)$$

according to previous results |12|.

Let us note, finally, that the MC (7) and the SC (11) detectors are canonical receivers: the general form and the decision process are not affected by the particular type of interference.

3. PERFORMANCE ANALYSIS OF THE SINGLE-CYCLE DETECTOR

An exact evaluation of the detector performance would require an extensive numerical computation or computer simulations . In fact, in order to calculate the false-alarm rate and the detection probability, the knowledge of the conditional (under both hypotheses) pdf's of the decision variable Z is necessary. Such pdf's are very cumbersome to evaluate and, moreover, even if a large-sample size assumption holds, unlike the known-signal case |7,10,11|, one cannot resort to the central limit theorem in that the rv's $R_g^\alpha(\cdot)$ are not mutually independent.

In order to circumvent what appears to be an intractable problem from a practical standpoint, but to still obtain useful information about the performance of the SC detector operating in non-Gaussian noise, the deflection |12| can be evaluated.

The squared deflection $D^2$, which is a measure of the output SNR in weak-signal cases, is defined by

$$D^2 \triangleq |E_1[Z] - E_0[Z]|^2 / \operatorname{var}(Z|H_0) \quad (15)$$

where $E_0[\cdot]$ and $E_1[\cdot]$ denote the expectation conditioned to $H_0$ and $H_1$, respectively, and $\operatorname{var}(Z|H_0)$ is the variance of the decision variable under $H_0$.

Since the rv's $n_i$ are iid, one has:

$$E_0[Z] = K_s^\alpha(0)\left\{E_0[g^2(y)] + E_0[g'(y)]\right\}\sum_{i=1}^{M} e^{j2\pi\alpha i} \quad (16)$$

Under the hypothesis $H_1$, by exploiting the weak-signal assumption and the independence among the rv's $s_i$ and $n_i$, the conditional expectation $E_1[Z]$ can be written as

$$E_1[Z] = E_0[Z] + \frac{1}{2} K_s^\alpha(0)\, F[f(y)]\sum_{i=1}^{M} K_s(i,i)e^{j2\pi\alpha i}$$

$$+ E_0^2[g^2(y)]\sum_{i,m=1}^{M} K_s(i,m)K_s^\alpha(i-m)e^{j\pi\alpha(i+m)} \quad (17)$$

where

$$F[f(y)] \triangleq E_0\{[f''(y)/f(y)]^2\} - 2E_0^2[g^2(y)] \quad (18)$$

The evaluation of $\operatorname{var}(Z|H_0)$ leads to

$$\operatorname{var}(Z|H_0) = M|K_s^\alpha(0)|^2 \operatorname{var}[g^2(y)+g'(y)|H_0]$$

$$+ 2M E_0^2[g^2(y)]\sum_{\substack{i=1-M\\ i\neq 0}}^{M-1} |K_s^\alpha(i)|^2 \quad (19)$$

By substituting (5) in (17) and by assuming the sample size M sufficiently large relative to the longest period $\alpha_{min}^{-1}$ of cyclostationarity ($M\alpha_{min}$>>1) and greater than the whidths of the cyclic autocorrelation functions $K_s^\alpha(i,m)$, from (15), (17) and (19) it follows that the squared deflection $D_{SC}^2$ of the SC detector is given by

$$D_{SC}^2 = \operatorname{var}(Z|H_0)/4 \quad (20)$$

If one assumes that the cyclostationary signal to be detected is embedded in Gaussian noise, from (20) it follows that

$$D_{SC_G}^2 = \frac{M}{2\sigma_n^4}\sum_{i=1-M}^{M-1} |K_s^\alpha(i)|^2 \quad (21)$$

according to the result reported in |13|.

In order to make a comparison between the performance of the SC detector and that of the LO receiver synthesized for a stationary signal (i.e., the detector operating according to (11) specialized for $\alpha$=0) we evaluate the asymptotic relative efficiency that can be expressed as ratio of the squared deflections. Therefore, from (20) it follows that

$$ARE_{SC,LO} = \frac{D_{SC}^2}{D_{LO}^2} = \frac{|K_s^\alpha(0)|^2\{F[f(y)] + 2E_0^2[g^2(y)]\tau_\alpha\}}{K_s^2(0)\ \{F[f(y)] + 2E_0^2[g^2(y)]\tau\}} \quad (22)$$

where

$$\tau_\alpha \triangleq \sum_{i=1-M}^{M-1} |K_s^\alpha(i)|^2/|K_s^\alpha(0)|^2 \quad (23)$$

is the width parameter of the cyclic autocorrelation function and $\tau$ is the corresponding parameter for $\alpha$=0.

The equations:

$$|K_s^\alpha(0)| \leq K_s(0) \quad (24)$$

$$\sum_{i=1-M}^{M-1} |K_s^\alpha(i)|^2 \leq \sum_{i=1-M}^{M-1} K_s^2(i) \quad (25)$$

derived by means of the fundamental inequality |14|

$$|S_s^\alpha(f)|^2 \leq S_s(f+\tfrac{1}{2}\alpha)\, S_s(f-\tfrac{1}{2}\alpha) \quad (26)$$

with $S_s^\alpha(\cdot)$ and $S_s(\cdot)$ denoting the cyclic and the conventional spectrum (respectively), allow us to state that

$$F[f(y)] \geq 0 \quad (27)$$

is a condition sufficient to assure that the SC detector, for any cycle frequency $\alpha$, cannot outperform the LO detector (i.e., $ARE_{SC,LO} \leq 1$). In the particular case of Gaussian noise, since $F[f(y)]$=0, from (22) it follows that the LO detector for stationary signals (also referred to as radiometric detector) has the largest possible deflection compared with all SC detectors.

Since the square-law detector is widely employed to detect random signals, it is useful to compare the performance of such a receiver with that of SC one.

The squared deflection $D_{SL}^2$ of the SL detector can be expressed as

$$D_{SL}^2 = \mathcal{E}^2\left\{M\left[\int_{-\infty}^{\infty} x^4 f(x)\,dx - \sigma_n^4\right]\right\}^{-1} \simeq MK_s^2(0)/\operatorname{var}(n_i^2) \quad (28)$$

where $\mathcal{E}$ denotes the signal energy and the approximation is accurate if M is sufficiently large to the longest period of cyclostationarity (i.e., $\alpha_{min}M$>>1). Therefore, from (20), (19), (3), (23) and (28) it follows that

$$ARE_{SC,SL} = \frac{D_{SC}^2}{D_{SL}^2} = \frac{1}{4}\operatorname{var}(n_i^2)\frac{|K_s^\alpha(0)|^2}{K_s^2(0)}$$

$$\cdot\{E_0[(f''(y)/f(y))^2] + 2(\tau_\alpha - 1)E_0^2[g^2(y)]\} \quad (29)$$

In the particular case of the SC detector with $\alpha$=0, it is interesting to note that (29) shows the $ARE_{SC,SL}$ as a sum of two terms: the former provides the $ARE_{SC,SL}$ on the assumption of white signals (i.e., $\tau$=1), the latter takes into account the correlation among the signal samples.

4. CONCLUSIONS

With reference to the detection of almost cyclostationary signals embedded in non-Gaussian noise, a locally optimum structure is derived. The LO detection statistic, which is a quadratic form, is expressed in terms of cyclic autocorrelation functions of the signal to be detected and cyclic correlograms of the signal obtained processing the received one by means of a ZMNL which depends only on the noise statistics.

Since the physical implementation of such multi--cycle detector could be quite involved, owing to the required phase estimation of the unknown signal, the so-called single-cycle detector is considered. It takes into account only one periodicity of the signal to be detected and, therefore, does not need phase estimate.

Since in weak-signal situations the deflection is a useful measure of the detection performance, its expression for the SC detector is derived. Moreover, in order to make a comparison between the performance of the SC detector and that of the LO receiver synthesized on the assumption of stationary signals in non-Gaussian noise, an expression of the asymptotic relative efficiency is provided. It is also carried out a condition on the noise statistic sufficient to assure that the SC detector cannot outperform the LO one.

Finally, since the square-law detector is often utilized in the applications, a comparison is made (again in terms of ARE) between such a detector and the SC one when they work in a non-Gaussian noise environment. It is shown that, in the particular case of stationary signals, $ARE_{SC,SL}$ is a sum of two contributions: the former provides the $ARE_{SC,SL}$ on the assumption of white signal; the latter takes into account the correlation among the signal samples.

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