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IMAGE RESTORATION FROM MAGNITUDE ONLY USING PROJECTION METHODS

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RESUME

Dans cette presentation nous etudions comments reconstruire une image en partant exclusivement de l´ampleur des transformations de Fourier. Nous utilisons un nouvel algorithme appellé la methode de projection generale. Cet algorithme peut detecter deux phenomenes pathologique: pieges et impasses qui semblent etres les solutions veritables mais en realité sont de fausses solutions. Des resultat sont presente´s a´l´appui.

SUMMARY

In this paper we study the problem of restoring an image from the magnitude of its Fourier transform. We use a new algorithm called the method of generalized projections. The algorithm can detect two pathological phenomena: traps and tunnels which appear as false solutions. Results are presented that illustrate correct restorations from magnitude information only.

1. INTRODUCTION

A fundamental idea in image restoration is that the signal or image to be restored, f, is known to lie in m given sets C_i, i=1,2,...,m where each of the sets represents a constraint on the image. Associated with each set C_i is a projection operator P_i, i=1,...,m. In general, for all sets -- not just convex ones -- we call $g \overset{\Delta}{=} P_i h$ the projection of h onto C_i if $g \in C_i$ and

$$||g-h|| = \underset{\text{all } y \in C_i}{\text{Min}} ||y-h|| \quad (1)$$

for i=1,2,...,m. $||g||$ denotes the norm of g.

1.1. Remarks

1) The projection as defined in Eq. (1) is a unique point if C_i is a convex set. When C_i is non-convex there may be more than one point that satisfies the definition of projection. However, in practice, we can usually find a procedure for uniquely choosing one of these points, normally through the demand of satisfying another condition. This eliminates the ambiguity that would otherwise result from non-singleton projection points. For instance in the restoration from magnitude problem, in projecting onto the set of functions with prescribed Fourier magnitude, the phase of the estimate at the n'th iteration uniquely defines the projection.

2) We assume that in all problems of interest there exists at least one point in C_i that is the projection of an arbitrary L_2 signal (L_2 is the space of square integrable functions).

A method for restoring f when all the sets are convex is given by the method of projection onto convex sets (POCS). The POCS algorithm [1]

$$f_{n+1} = T_1 T_2 \dots T_m f_n, \quad f_0 \text{ arbitrary}, \quad (2)$$

where

$$T_i \overset{\Delta}{=} L = \lambda_i (P_i - 1), \quad i=1,2,\dots,m, \quad (3)$$

is known to converge to a point in $C_0 = \bigcap_{i=1}^{m} C_i$ provided that C_0 is not empty and $0<\lambda_i<2.0$.

2. RESTORATION BY GENERALIZED PROJECTIONS

If one or more of the set C_i is non-convex then the convergence of the algorithm given by Eq. (2) is not guaranteed. Nevertheless the algorithm for m=2, i.e.,

$$f_{n+1} = T_1 T_2 f_n, \quad f_0 \text{ arbitrary} \quad (4)$$

have some properties that make it useful for image restoration. Since convergence of Eq. (4) is not assured we need some measure that will allow us to gauge the performance of the algorithm during the iteration process. The performance measure is needed also as an indicator for when parameter changes are required in the algorithm in order to improve its performance. Such a measure is provided by the summed-distance-error (SDE) defined as follows. For any vector g in L_2, the SDE denoted by J(g), is given by

$$J(g) = ||P_1 g - g|| + ||P_2 g - g||. \quad (5)$$

The SDE is the sum of distances from g to the two sets C_1 and C_2. In general $J(g) \geq 0$ and $J(g)=0$ if and only if $g \in C_1 \cap C_2$. When J(g) is small the signal is "close" to satisfying all the constraints imposed on it; when it is large the signal is far from satisfying the *a priori* constraints. The main property of the recursion given by Eq. (4) is the set distance reduction property described in the following theorem.

2.1. Theorem

The recursion given by Eq. (4) has the property that

$$J(f_{n+1}) \leq J(T_2 f_n) \leq J(f_n) \quad (6)$$

for every λ_1 and λ_2 that satisfy

$$0 \leq \lambda_i \leq A(f_n), \quad i=1,2, \quad (7)$$

Here $A(f_n)$ (not given here because of its lengthy form) depends only on the latest estimate f_n and on the operators P_1, P_2. The proof of this theorem is given in Ref. [2].

2.2. Remarks

1) It can easily be shown that the range of λ_i in Eq. (7) always includes the interval [0,1].

2) The set distance reduction property described by Eq.(6) does not extend in general to an algorithm such as in Eq. (2) with m > 2. [See [2] for a counter example].

3) Despite the fact that the theorem is not valid for m > 2 the algorithm given by Eq. (4) is not

especially restrictive in practice. This is because we can often combine those properties of the signal which are easily expressed in the space domain to one set C_1 whose associated projection operator P_1 can be calculated without too much effort. Similarly the properties of the signal which are easily expressed in the transform domain can be combined into a second set C_2 and the corresponding projection operator P_2 can again be calculated without too much effort.

4) The algorithm given by Eq. (4) can be optimized with respect to λ_1 and λ_2 on a per-step or per-cycle basis where the SDE, $J(f_n)$, is used as a criterion for minimization. More about this appears later in this paper in connection with the RFM problem.

5) The pathological behavior sometimes exhibited by the algorithm of Eq. (4) can be explained by the existence of: (i) fixed points of the operator P_1P_2 (g is a fixed point of P_1P_2 if $g=P_1P_2g$) which are not valid solutions. We call these points traps and they occur only when non-convex sets are involved; and (ii) tunnels in which the solution is approached so slowly that for all practical purposes the algorithm has ceased functioning. Traps and tunnels are illustrated in Fig. (1).

3. THE RESTORATION FROM MAGNITUDE PROBLEM

3.1. General

The two sets involved in the restoration from magnitude (RFM) problem are: C_1 the set of space-limited functions (any two level amplitude constraints (i.e., $a \leq f \leq b$ can easily be added) and C_2 the set of all functions which have a Fourier transform magnitude equal to some real positive prescribed function $M(\omega)$. Thus

$$C_1 = \{g(x): \ g(x)=0 \text{ for } |x|>a\} \qquad (8)$$

$$C_2 = \{g(x)\leftrightarrow G(\omega): |G(\omega)|=M(\omega) \text{ for all } \omega\}. \qquad (9)$$

It can easily be verified that C_1 is convex and that C_2 is non-convex. The projections P_1 and P_2 onto C_1 and C_2 are respectively given by

$$P_1 g = \begin{matrix} g(x), & |x|<a \\ 0 \quad , & |x|\geq a \end{matrix} \qquad (10)$$

and

$$P_2 g \leftrightarrow M(\omega)e^{j\phi(\omega)} \qquad (11)$$

where $\phi(\omega)$ is the phase of $G(\omega)$. P_2g is uniquely defined by Eq. (11) although C_2 is non-convex.

The general restoration algorithm is given by Eq. (4) with T_i, i=1,2 defined in Eq. (3): $T_i=1+\lambda_i(P_i-1)$. This algorithm has the property of set-distance reduction, or the property that $\{J(f_n)\}_{n=0}^{\infty}$ is a non-increasing sequence, for (at least) those values of λ_1,λ_2 that satisfy the inequality (6).

3.2. Fundamental Remark

When $\lambda_1=\lambda_2=1$, Eq. (4), with P_1 and P_2 as defined in Eqs. (10) and (11) reduces to the Gerchberg-Saxton algorithm [3] and the property that $\{J(f_n)\}_{n=0}^{\infty}$ is a non-increasing sequence becomes equivalent to the non-increasing error property described by Fienup [4]. Also, it is readily shown that Fienup's [4] output-output algorithm is equivalent to a special form of the set-distance reduction algorithm.

3.3 Optimization of the Relaxation Parameters (RP) $\lambda_{i=1,2}$

It can be shown [2] that an optimum (per-cycle) algorithm when C_1 is a linear subspace is given by

$$f_{n+1}=P_1T_2f_n, \quad f_0 \text{ arbitrary.} \qquad (12)$$

This algorithm is generally near-optimum when C_1 is not a linear subspace. The optimal value of λ_2 in Eq. (12) can be found by a search which is relatively fast when C_1 is linear.

3.4. Traps and Tunnels in the RFM Problem

The algorithm of Eq. (12) is caught in a trap if $f_n=P_1T_2f_n$ yet f_n is not one of the valid solutions. Recall that a valid solution satisfies $f \in C_1 \cap C_2$. When the algorithm enters a tunnel the change from iteration to iteration is negligible, i.e., f_n $P_1T_2f_n$. In the special case of restoration from magnitude the existence of traps cannot be easily demonstrated theoretically. However, the existence of traps is supported by the real difficulties one encounters in restoring some signals or images from their magnitudes.

The following two observations are of great practical importance regarding traps and tunnels. (i) The SDE can be used to detect traps. By this we mean that being in a trap is equivalent to no change in $J(f_n)$ from iteration to iteration. (ii) When P_1 is a linear operator (as in Eq. (10)) then a correct solution f lies in a hyperplane orthogonal to the vector $P_2f_n-f_n$, i.e., f_n-f is orthogonal to $P_2f_n-f_n$. If P_1 is not linear this is only approximately true.

4. EXPERIMENTAL RESULTS

In this section we describe the results of restorations from magnitude for two synthetic images: IMAGE 1 and IMAGE 2 given in Fig. 2 and 3 respectively. The non-zero portion of each image is confined to a region of 30x30 pixels in the center of a total field of 64x64 pixels. IMAGE 1 is composed of six gray levels with minimum level 0 and maximum level 1. IMAGE 2 is a two level binary image of 0's and 1's only. For the restoration experiments we used the four algorithms described below:

1) The Gerchberg-Saxton (GS) algorithm ($f_{n+1}=P_1P_2f_n$) with C_1, C_2, P_1 and P_2 as given by Eqs. (8-11);

2) The same algorithm as in (1) except with C_{1L} and P_{1L} replacing C_1 and P_1. C_{1L} is a subset of C_1 that includes a two-level amplitude constraint:

$$C_{1L}=\{g(x):\ g(x)=0 \text{ for } |x|>a \text{ and } 0 \le g(x) \le 1 \text{ for } |x|\le a\}, \tag{13}$$

and P_{1L} is the projection operator that projects onto C_{1L}

3) The relaxed projections algorithm i.e., the algorithm using optimum relaxation parameters as given by Eq. (12) with C_1, C_2, P_1 and P_2 as before;

4) The same algorithm as in (3) with C_{1L} and P_{1L} replacing C_1 and P_1.

In the relaxed algorithms (3) and (4) a search for the optimal value, λ_{2m}, of λ_2 was made.

Figure 4 gives the restored images after 30 iterations for the above four algorithms with initial points $f_0=0$. Panels a and b of Fig. 4 result from algorithms (1) and (3) respectively (i.e., without using the two-level constraint) and give poor but recognizable images. The positive background in Figs. 4a and 4b (instead of zero) is to the negative portions in the restored images which make it necessary for the sake of display to shift the level of the images upward. Figures 4c and 4d result from algorithms (2) and (4) respectively (i.e., with the two-level constraint) and show much better restorations. Indeed algorithm (4) yields a result indistinguishable from the original. Note the coordinate reversal of the restored images: if f(x,y) is the original image then f(-x,-y) is the coordinate-reversed image and has the same magnitude function. This coordinates reversal can be seen in the following figures as well. Figure 4 shows the significant improvement when relaxation parameters are used. The example in Fig. 4 also shows the importance of projecting onto the sets of functions satisfying the two-level constraints for good restorations using a relatively small number of iterations. Figure 5 shows the restorations of IMAGE 2 for the same four algorithms as before and for the same initial point. In Fig. 5 the restored images for the four algorithms are given after 100 iterations. The very poor restorations together with the very small changes observed in $J(f_n)$ after a certain number of iterations is symptomatic of the condition whereby the algorithm is either near a trap or is approaching the correct solution through a tunnel.

We also wanted to demonstrate the effect of the starting point. Figure 6 shows the results of the restoration of IMAGE 2 with an initial point $f_{01}(x,y)$ defined as follows: Let S_p be the support of f(x,y) i.e., 30x30 points in the center of the total field of 64x64 points. $f_{01}(x,y)=0.72$ for a region of 20x20 points in the center of S_p and $f_{01}(x,y)=0.36$ for the remaining points of S_p. Figure 6 shows the restored images for this case without (a) and with (b) the use of relaxation parameters.

Here we clearly see the importance of the initial point for rapid restoration. With $f_{01}(x,y)$ we get a recognizable image with pure projections and a very good image with the use of relaxation parameters, after only 40 iterations.

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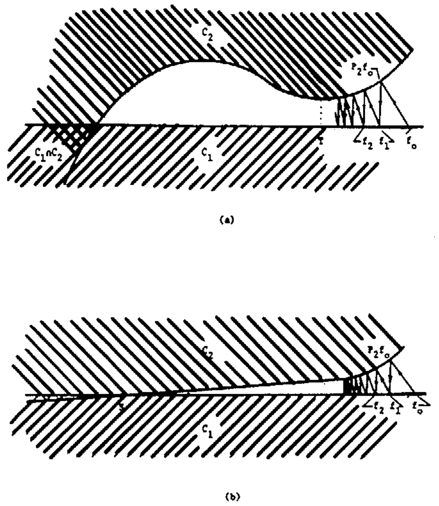

Fig. 1 Illustration of a trap and tunnel for an algorithm of the form $f_{n+1}=P_1P_2f_n$

(a) Starting at the point f_o the sequence $\{f_n\}$ converges to a trap point T while the true solution must belong to $C_1 \cap C_2$.

(b) Starting at the point f_o the algorithm enters into a long tunnel towards the solution at the point S.

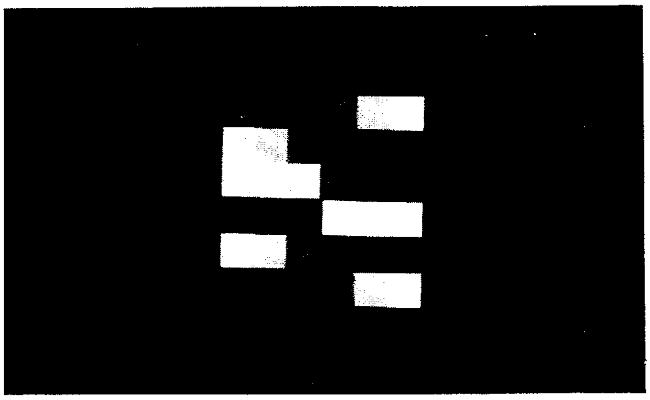

Fig. 2 Original Image IMAGE 1.

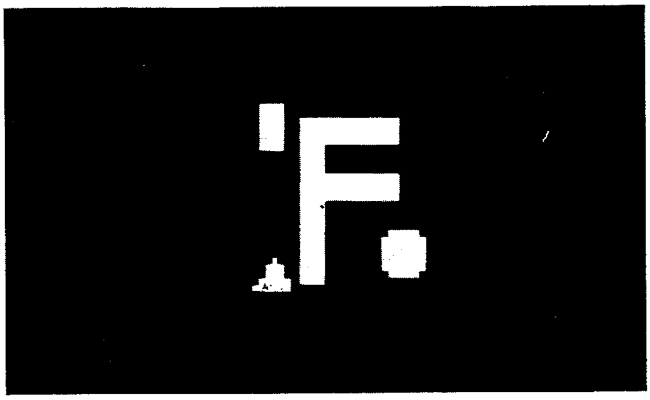

Fig. 3 Original Image IMAGE 2.

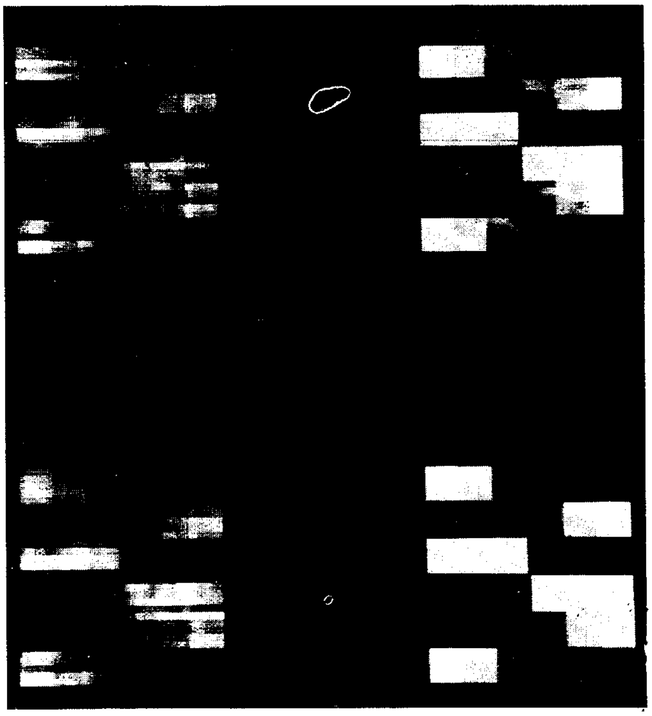

Fig. 4 Restoration of IMAGE 1 After 30 Iterations With Initial Point $f_o=0$.

(a) (Upper left). Restoration by $f_{n+1}=P_1P_2f_n$.

(b) (Lower left). Restoration by $f_{n+1}=P_1T_2f_n$.

(c) (Upper right). Restoration by $f_{n+1}=P_{1L}P_2f_n$.

(d) (Lower right). Restoration by $f_{n+1}=P_{1L}T_2f_n$.

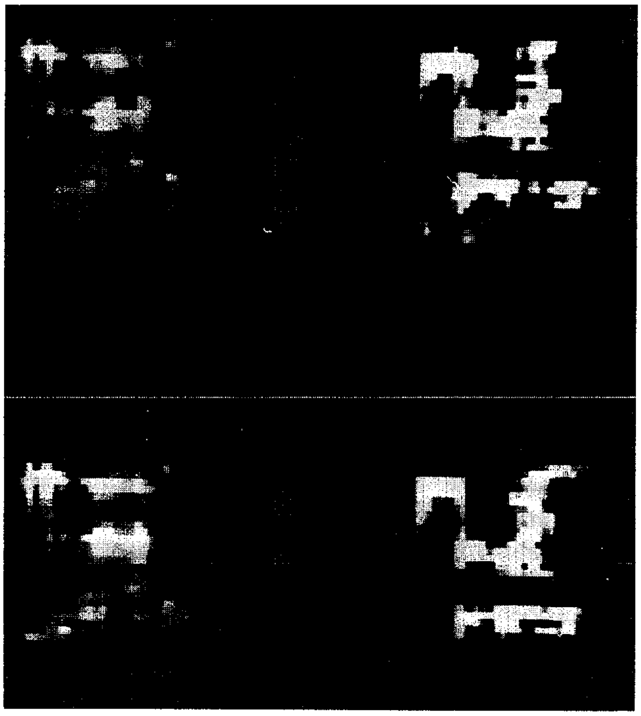

Fig. 5 Restoration of Image 2 After 100 Iterations With $f_o=0$.

(a) (Upper left). By $f_{n+1}=P_1P_2f_n$.

(b) (Lower left). By $f_{n+1}=P_1T_2F_n$.

(c) (Upper right). By $f_{n+1}=P_{1L}P_2f_n$.

(d) (Lower right). By $f_{n+1}=P_{1L}T_2f_n$.

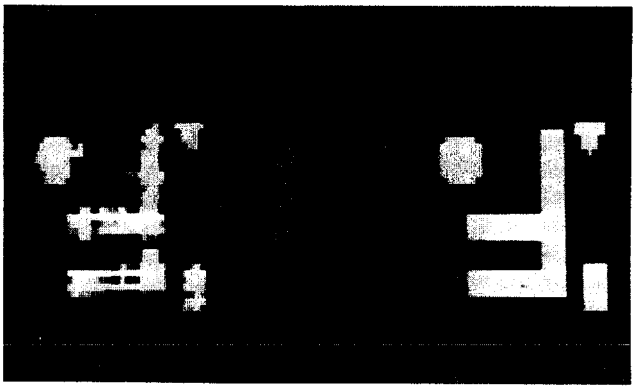

Fig. 6 Restoration of IMAGE 2 After 40 Iterations With Initial Point $f_{01}(x,y)=0.72$ for 20x20 Pixels in the Center of the 30x30 Pixels of the Support, S_p, of the Function f. $f_{01}(x,y)=0.36$ for All Other Points of S_p.

(a) (Left). By $f_{n+1}=P_{1L}P_2f_n$.

(b) (Right). By $f_{n+1}=P_{1L}T_2f_n$.