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IMAGE RECONSTRUCTION FROM PARTIAL INFORMATION - ZEROS AND SINE WAVE CROSSINGS

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RESUME

SUMMARY

The sine-wave-crossing approach overcomes the problem of instability inherent in signal reconstruction from estimated zero crossing locations. The theoretical framework of Logan provides several stable reconstruction algorithms for one-dimensional signals. Extending these results to sine-wave-crossing contours, we reconstruct images by application of an interpolation algorithm similar to the one used in recovering a signal sampled uniformly at Nyquist rate. A sampled version of Logan's theorem is implemented, investigating both theoretically and computationally the effects of zero-location quantization on signal reconstruction. Adopting a stochastic approach, bounds on the reconstruction error of a bandlimited and an almost bandlimited signals are derived. The bounds indicate the effects of number of quantization levels, of amplitude and frequency of the added sine wave and of out-of-band energy, on the resultant m.s.e.

Image reconstruction from partial information - zeros and sine wave crossings

. Introduction

The problem of image representation by partial information has recently been studied extensively in both the spatial and frequency domains. In the frequency domain it was shown that either amplitude, phase or even one bit of phase will suffice for unique reconstruction within some parameter (for review see [1]). Alternatively one can approach the problem in the space (or time) domain employing various sampling schemes, or more specifically sets of points at which the signal crosses a given reference signal. The proposition that images can be recovered from zero crossings has attracted a great deal of interest [2]. This became in particular attractive in view of Logan's results showing that under certain conditions a bandpass signal can be uniquely represented by its zero crossings [3]. Logan however didn't provide a reconstruction algorithm. Extending his work to Band Pass - Band Limited two-dimensional signals, Zeevi and Rotem [4] subsequently succeeded in reconstructing images from zero crossings. Exploiting the duality of the Fourier-Stieltjes transform, Shitz and Zeevi [5] based their approach on Logan's results and rederived most of the theorems concerning one-dimensional signal reconstruction from partial information in the frequency domain. Using the properties of two-dimensional polynomials [6], or entire functions of exponential type in two variables [7], Curtis et al showed that indeed it is likely that a two-dimensional signal can be uniquely represented by its zero crossings. However in neither of these studies was a theoretically stable reconstruction algorithm introduced. (Stability is interpreted here as the sensitivity to the accuracy of crossing locations and to frequency band constraints)

Dealing with one-dimensional signals, Bar-David [8] and Logan [9,10] showed that by adding a sine wave satisfying certain conditions, one can recover the original signal from its sine wave crossings. The latter work accomplished a complete theoretical framework for the sine-wave-crossing problem, providing several stable reconstruction algorithms. In this work we extend some results to two-dimensional signals, reconstructing images from samples of sine-wave-crossing contours. Considering the error introduced by finite approximation of the crossing locations, we adopt a stochastic approach and derive an upper bound on the m.s.e. We also consider the error resulting from an almost bandlimited signal.

II. Zeros and Sine-Wave-Crossings - One-Dimensional Signals

A sampling set for signals bandlimited to $[-W, W]$ should have a density of no less than π/W. This implies that the signal is represented by its samples taken at a rate greater than Nyquist. In considering the representation of a bandlimited signal by its zero or sine-wave crossings, one should specify conditions assuring the existence of a sampling set formed by crossing locations. Lowpass signals, bandlimited to $[-W, W]$,may have at most zero density of π/W, which does not satisfy (except in degenerate cases) the above condition. One way of forming ,under these circumstances, a representation (sampling) set is by adding a high frequency sine-wave component, resulting in a set of crossings with density equal to that of the zero crossings of the sine wave alone. Alternatively, one can satisfy the required sampling rate by shifting the frequency band, transforming the original lowpass into a bandpass signal. In the latter case the minimum crossing rate can be specified; it is determined by the lowest frequency in the band. With the proper constraint of less than one octave in bandwidth, the attainable rate can be greater than the required one. Logan [3] proved that under certain conditions the zero crossings form a sampling set. The conditions are as follows: First, the bandwidth has to be less than one octave (assuring at least a Nyquist rate). Second, only simple real zeros of the signal are permitted to be in common with its Hilbert transform (this eliminates, for example, the possibility of sampling carrier zeros of amplitude modulated signals). Logan's theory does not provide, however, a reconstruction scheme. Further, there seems to be an inherent problem of stability with respect to both the bandwidth limitations and common zero condition. We have therefore adopted the sine wave crossing solution in our work with images.

Bar-David [8] and Logan [9,10] independently derived theorems concerning the problem of sine wave crossing. The theorems state that a signal $s(x)$ bandlimited to $[-W_b, W_b]$ (in some sense), can be recovered from the set of crossing points $\{x_i\}$ of a given sine wave $A\cos W_c x$ satisfying $W_c > W_b$ and $|s(x)| < A \ \forall \ x$ (it is sufficient to require $(-1)^i s(i\pi/W_c) < A \ \forall \ i$ instead of $|s(x)| < A \ \forall \ x$; see [9]). This is treated as a zero crossing problem since we may write

$$f(x) = s(x) + A\cos W_c x \tag{1}$$

The signal $f(x)$ has only real simple zeros, each one being situated between two consecutive extrema of the sinusoid

$$f(x_i) = 0 \ ; \ where \ x_i \in \left[i\frac{\pi}{W_c}, (i+1)\frac{\pi}{W_c} \right] \tag{2}$$

Since the zeros occur at Nyquist rate, they form a sampling set for $f(x)$ or, more precisely, for $s(x)$.

Applying the factorization formula [4]

$$f(x) = f(0)\prod_{i=-\infty}^{\infty}\left(1-\frac{x}{x_i}\right) \ ; \ where \ f(0) \neq 0 \tag{3}$$

one obtains a mathematical algorithm for reconstructing $f(x)$ from its set of zeros $\{x_i\}$ within a multiplicative constant. The product (3) converges provided the set of zeros is correctly ordered, i.e., $|x_{i+1}| > |x_i|$. This formula is however impractical for the following reasons: First, since $f(x)$ (and therefore $s(x)$ as well) should be strictly bandlimited, the product may not converge even if $\{x_i\}$ are the zeros of an almost bandlimited signal. Second, knowledge of all the zeros and their exact locations is required.

Alternatively, a fundamental function $h(x)$ is associated with the zeros $\{x_i\}$ of $f(x)$ [9,10]

$$h(x) = J(x) - W_c \tag{4}$$

where $J(x)$ is a count function increasing by π at each zero of $f(x)$. The function $h(x)$ is bounded by $\pi, |h(x)| < \pi$, and vanishes over the interval $(-\lambda, \lambda)$, where $\lambda = W_c - W_b$ is the gap frequency. This high pass property of $h(x)$ allows the localization of the infinite product (3), which means that an estimate of $f(x)$ can be made based on the zeros in the window $(x-\Delta x, x+\Delta x)$. This estimation process results in an error of the order $exp(-\lambda\Delta x)$. This method is stable since small perturbation of the zeros produce small error in the reconstructed signal. Further it converges even if a practical signal, not strictly bandlimited, is used. The error then is proportional to the signal energy outside the interval $[-W_b, W_b]$.

In the case where $W_c > 3W_b$ one has only to low-pass filter the nonuniform samples of $s(x)$ as if they were uniform [10]. The reconstruction formula

$$s(x) = -\sum_{i=-\infty}^{\infty}\cos W_c x_i \frac{\sin W_b(x-x_i)}{W_c(x-x_i)} \tag{5}$$

is derived in this case using the high pass property of the derivative of the fundamental function $h(x)$

$$h'(x) = \pi\sum_{i=-\infty}^{\infty}\delta(x-x_i) - W_c \tag{6}$$

In our work with images we adopt this algorithm because of its simplicity.

III. Image Reconstruction from Sine-Wave-Crossings

An extension of theorems concerning one-dimensional signals to two dimensions (or multidimensions) is not, in general, a straightforward task. Further, any attempt to extend results on one-dimensional zeros or sine-wave crossings has to confront the basic problem that two-dimensional signals do not have any isolated zeros, and therefore are nonfactorable, in general. (Specificly, entire functions of exponential type (EFET) in two complex variables are not a ring of factorization, unlike EFET of one complex variable or polynomials of fixed degree in one or two variables). In two dimensions the crossings are in the form of contours consisting of an infinite number of points. It is therefore likely that in zero crossing contours there exists sufficient information as required (and even more) for signal representation and reconstruction. Of the infinite number of crossing points along the contours a finite number of samples will suffice. Some recent work of Zeevi and Rotem [4] and of Curtis et al [6,7] was devoted to this problem, resulting in successful reconstruction of images from their zero crossing contours.

Image reconstruction from partial information - zeros and sine wave crossings

Considering two-dimensional signals (images), it is not in general guaranteed that all the signals satisfy the conditions required for reconstruction from level crossings [4,6,7]. Although it can be argued that the conditions are not too restrictive. However, one can not claim that two-dimensional signals, which do not satisfy the required conditions, are not important (e.g. circular symmetric etc. [7]). Therefore it seems reasonable to impose zero crossings by (for example) adding a known signal such as sine-wave, a technique previously applied in studies of one dimensional signals [8-10]. This leads to the proposition that two-dimensional signals (especially images) may be recovered from sine-wave crossing contours. The two-dimensional signal is treated as a parametric set of one-dimensional signals (with the parameter representing the second independent variable). The one-dimensional signal, obtained for each value of the parameter, satisfys Logan's conditions.

Let $s(x,y)$ be a real-valued function bandlimited to $[-W_x, W_x] \times [-W_y, W_y]$ and bounded by A; $|s(x,y)| < A \ \forall \ x,y$. Let $\{l_i(x,y)\}$ be the set of contours defined by

$$f(x,y) = s(x,y) + A\cos(W_{cx}X + W_{cy}Y) = 0 \; ; \qquad (7)$$

$$where\ W_{cx} > W_x,\ W_{cy} > W_y .$$

The extended theorem in two dimensions states that $s(x,y)$ can be uniquely recovered from the set $\{l_i(x,y)\}$. In fact, a smaller set, formed by sampling of $\{l_i(x,y)\}$, is sufficient for unique reconstruction.

The following method is used to sample and reconstruct the images (Fig. 1). A uniformly sampled sine-wave $A\cos(W_{cx} m\Delta x)$ is added to a sampled image $s(m\Delta x, n\Delta y)$, where $\Delta x = \Delta y = \pi / W_{cx}$. (The median of image intensity is subtracted from the signal to generate both positive and negative image values.) The sine-wave satisfys $W_{cx} > 3W_x$ and $A > (-1)^i s(i\pi / W_{cx}, n\Delta y) \ \forall \ i,n$. Zero crossing locations are estimated using the points where the resultant signal

$$f(m\Delta x', n\Delta y) = s(m\Delta x', n\Delta y) + A\cos(W_{cx} m\Delta x') \qquad (8)$$

changes sign, with $\Delta x' = 2^{-n}\Delta x$. The required accuracy is achieved by an interpolation (oversampling) of the order 2^n. The points $(i\Delta x', n\Delta y)$, at which the signal $f(m\Delta x', n\Delta y)$ changes sign, are used to generate the signal $A\cos(W_{cx} i\Delta x')$ for each row n. The sampled sine-wave, at the points $(i\Delta x', n\Delta y)$, is low pass filtered (along the x axis) according to (5), to obtain the reconstructed image (Fig. 1).

The normalized m.s.e of the reconstructed images (Fig. 2) behaves like 2^{-2n}, where 2^n is the oversampling rate. The error decreases as the "average power" increases, since effectively there are more quantization levels.

IV. Reconstruction Error - Derivation of a Bound

Implementing a digital version requires quantization of the spatial axis. This gives rise to a reconstruction error due to inaccurate estimation of zeros' location. In general, the derivation of an error formula is complicated, and only a bound can be derived using some assumptions.

In order to derive a bound, we adopt a stochastic approach, assuming that:

(1) The signal $s(x)$ is a stationary lowpass process.

(2) The resultant error $e(x) = \hat{s}(x) - s(x)$ is a stationary lowpass process (with a finite support of less than $[-W_c, W_c]$), where $\hat{s}(x)$ denotes the signal recovered from the set of sampling points.

(3) The position errors $\{\varepsilon_i\}$ associated with crossing locations $\{x_i\}$ are distributed uniformly $s.t.\ |\varepsilon_i| < 2^{-(n+1)}\Delta x$, where $\Delta x = \pi / W_c$ is the Nyquist interval, and 2^n is the number of quantization levels.

(4) The position errors $\{\varepsilon_i\}$ are statistically independent of $s(x_i)$.

It can be shown [11] that under the above assumptions the normalized mean square error $E[e^2(x)]/E[s^2(x)]$ is bounded by

$$\frac{E[e^2(x)]}{E[s^2(x)]} \le W_c^2 E(\varepsilon_i^2) \left[\left(\frac{A^2}{E[s^2(x)]} - 1 \right) + \frac{W_b^2}{W_c^2} \right] \qquad (9)$$

indicating, as expected, the effects of the number of quantization levels, and of amplitude and frequency of the added sine wave, on the resultant error. The m.s.e behaves like 2^{-2n} (where 2^n is the number of spatial quantization levels). It is similar ,in a way, to the m.s.e resulting from amplitude quantization assuming uniform distribution (Fig. 2). Indeed images, as those depicted in (Fig. 1), are reminiscent of those obtained at various numbers of amplitude quantization levels. The above formula for the bound is valid for any reconstruction scheme, provided that Logan's conditions [9,10] and the above assumptions are satisfied.

Applying the algorithm to practical signals (which are in general not strictly bandlimited) results in a reconstruction error. An almost bandlimited signal $s_p(x)$ is a signal which can be approximated by its bandlimited component $s_l(x)$ (with finite support over $[-W_b, W_b]$), and which satisfys the condition that the energy $||s_p(x) - s_l(x)||^2$ outside the band $[-W_b, W_b]$ produces small shifts in zero (crossing) locations of the bandlimited signal $s_l(x)$. Hence, combining the above with equation (9), and using a lower bound on the expected variations of a bandlimited signal, provides a bound on the error resulting from an almost bandlimited signal. The bound for this type of error is [11]:

$$\frac{E[e^2(x)]}{E[s_l^2(x)]} \le \frac{\left[\left(\frac{A^2}{E[s_l^2(x)]} - 1 \right) + \frac{W_b^2}{W_c^2} \right]}{\left[\left(\frac{A^2}{E[s_l^2(x)]} - 1 \right) + \frac{4W_{eff}^2}{\pi^2 W_c^2} \right]} \frac{E[s_h^2(x)]}{E[s_l^2(x)]} \qquad (10)$$

where the effective bandwidth of the signal is defined by

$$W_{eff}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S(\omega) d\omega}{\int_{-\infty}^{\infty} S(\omega) d\omega}$$

with $S(\omega)$ being the power spectrum of the signal s_l, and s_h the highpass component

$$s_h = s_p - s_l$$

Comparison of computed and predicted errors due to out-of-band energy confirms the validity of the above formula.

Another type of reconstruction error results when a finite (truncated) series (5) is used. This error is the same, in principal, as the one resulting in recovery of a signal from its uniform samples using a finite series.

V. Discussion

We have shown that two-dimensional signals (and obviously one-dimensional signals) are recoverable from their sine-wave-crossings using digital methods. The algorithm presented is stable in the sense that a small error in zero locations results in a relatively small error in the recovered signal. Further, unlike other schemes, the algorithm can be applied to an almost band-limited signal, assuring convergence. The algorithm is practical because only a finite number of zeros are required in order to get a good estimation of the signal. The error, due to sampling the zeros in finite resolution, is affected by the amplitude and frequency of the added sine-wave relative to the amplitude and frequency of the original signal. The m.s.e behaves much like that established in the case of amplitude quantization assuming uniform distribution. The bound derived seems to match the experimental results (Fig. 2). This means that a finite number of bits are needed for coding zero crossing information thereby specifying a signal to finite precision, which is similar to that achieved in amplitude quantization.

The representation of a signal by its zeros does not require a wide dynamic range and it is preserved under various sorts of memoryless nonlinear operation. However, knowledge of the exact zero locations is required for error-free recovery. Sampling zeros at high resolution means wide bandwidth, while there is no need for wide dynamic range as required by a uniform sampling scheme. Hence this method viewed as a trade-off between the dynamic range and bandwidth.

We note that it is possible to use other of Logan's reconstruction schemes [9,10], but the easist to implement is the one introduced in this paper, requiring only lowpass filtering of the samples. The analytic upper bound on the reconstruction error, due to inaccuracy in crossing-locations, is independent of any recovery scheme under the mentioned assumptions.

The image reconstruction algorithm presented here is one-dimensional in nature and does not take advantage of inherent two-dimensional properties. The issue of two-dimensional nonuniform sampling and reconstruction is considered elsewhere [see 11].

References

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(a). (b). (d).

Fig. 1 - (a) Original image (LENA) used in processing and reconstruction by the sine-wave-crossing technique (256x256 pixels lowpass filtered to 42x42). (b), (c) and (d) Images reconstructed from zeros oversampled at 4, 16 and 64 times (n=2, 4 and 6) the Nyquist rate, respectively.

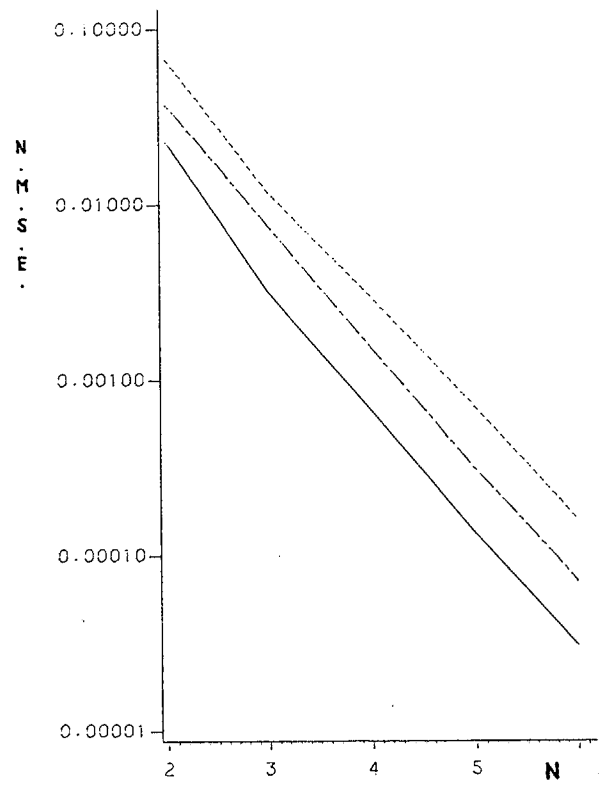

Fig. 2a - Normalized m.s.e. versus oversampling rate of 2^n computed for three different reconstructed images. The "average power" $E[s^2(x,y)]$ for each image is :
(—·) CLOCK - 33560; (-—) LENA - 16222; (----) GIRL - 6766;

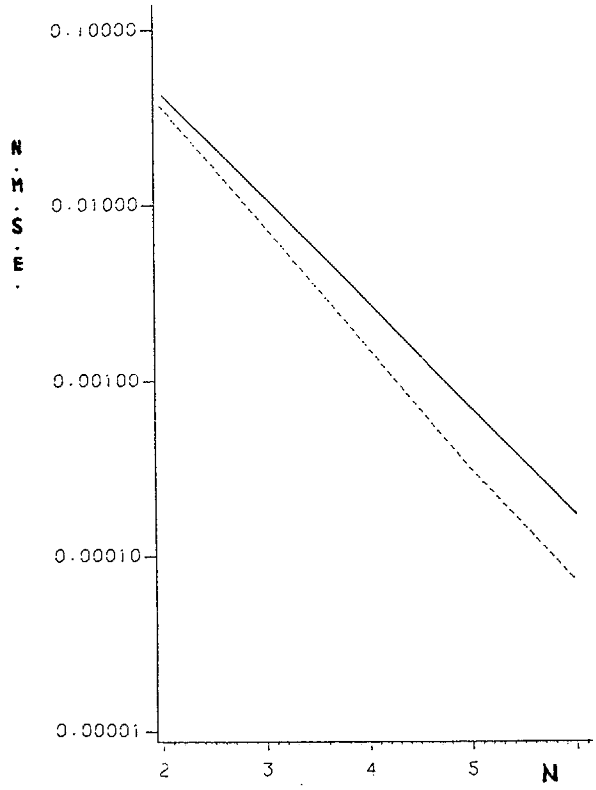

Fig. 2b - A comparison of the theoretical upper bound (eq. 9) (—) and the normalized m.s.e. computed for the reconstructed LENA image (----).