DIXIEME COLLOQUE SUR LE TRAITEMENT DU SIGNAL ET SES APPLICATIONS

NICE du 20 au 24 MAI 1985

ESTIMATION OF NOISY SINUSOIDS FREQUENCIES BY THE GENERALIZED LEAST SQUARES METHOD

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RESUME

Dans cet article, on propose l'application de la méthode des moindres carrés généralisée a l'estimation de la fréqence des sinusoides dans le bruit coloré additif. Dans l'analyse théorique on définit les expressions pour le limite en probabilité des estimations des parameters du model du signal, ainsi que pour la covariance de l'erreur d'estimation asymptotique. Les resultats obtenus permettent de construire un algorithme efficace pour les applications pratiques. L'analyse experimentale démontre que l'algorithme proposé donne des résultats précis mêmedans le cas des rapports signal sur bruit très petits.

SUMMARY

In this paper the application of the generalized least-squares method to the estimation of frequencies of sinusoids in additive colored is proposed. In the analysis expressions for both the probability limit of the estimates of signal model parameters and the asymptotic estimation error covariance are defined. The obtained results enable the construction of an algorithm efficient in practical applications. The experimental analysis shows that the proposed algorithm provides accurate results even in the case of very low signal-to-noise ratios.

INTRODUCTION

Estimation of frequencies of sinusoids in additive noise is one of very important problemsis digital signal processing. A large number of recently published papers have been devoted to this problem. Diverse solutions have been proposed, starting from different assumptions and using different methodologies (see e.g. |1|). An important class of frequency estimation algorithms is based on the parametric representation of the signal model in the transfer function form |2|. Such an approach allows the application of methods initially developed within control theory and system identification. It has been shown, for example, that the least-squares, instrumental variable maximum likelihood and iterative inverse filtering methods can be successfully applied even in the case of low SNR ratios and small data sets, e.g.|1,2,3,4,5|.

This paper discusses the application of the generalized least-squares method to the estimation of frequencies of sinusoids in colored noise. The theoretical analysis starts from the expressions for the asymptotic bias and error covariance in the case of the classical least-squares method. This methodology is extended, through a detailed derivation, to the generalized least-squares method, in which the asymptotic bias, characteristic for the least-squares method, is removed by adequate measurement filtering. The obtained results valid for one sinusoid in white noise, provide a deeper insight into the basic mechanisms of the method and relationships between variables and parameters involved. As a result, an algorithm of the generalized-least-squares type, efficiently applicable in practice, is derived. It possesses computational advantages over some popular versions of the maximum likelihood method. Experimental results clarify its characteristic properties given in comparison with the maximum likelihood and instrumental variable methods.

PROBLEM STATEMENT

Many frequency estimating algorithms are based on the following model which is valid for a signal $y(i)$ composed of a sum of n sinusoids $x(i)=\sum_{m=1}^{n} A_m \cos(\omega_m i+\phi_m)$ and additive stationary, zero-mean Gaussian noise $\varepsilon(i)$

$$A(z^{-1})y(i) = A(z^{-1})\varepsilon(i) \qquad (2.1)$$

Here $A(z^{-1}) = 1+a_1z^{-1}+\ldots+a_{2n}z^{-2n}$ is a polynomial in the backward shift operator, its coefficients beeing symmetric $a_j=a_{2n-j}$ for $j=0,1,\ldots,n$, $a_o = 1$, and its roots lying on the unit circle in the z-plane in complex conjugate pairs. The roots' angles correspond to n sinusoids' frequencies. In this way the problem of estimating the unknown frequencies becomes the problem of estimating parameters $A^T=[a_1a_2\ldots a_{2n}]$ or just $A^{*T} = [a_1a_2\ldots a_n]$ |1|.

Let the additive noise $\varepsilon(i)$ be represented by an ARMA model $\varepsilon(i) = (B(z^{-1}) / D(z^{-1}))\xi(i)$, where $\xi(i)$ is a white zero-mean Gaussian noise with variance σ_ξ^2 and $B(z^{-1})=1+b_1z^{-1}+\ldots+b_pz^{-p}$, $D(z^{-1})=1+d_1z^{-1}+\ldots+d_qz^{-q}$.

Assume that N signal measurements are available. Then the signal model can be written as

$$V_y(N) = -M_y(N)A + V_e(N) \qquad (2.2)$$

where $V_y^T(N) = [y(2n+1)y(2n+2)\ldots y(N)]$,

$$M_y(N) = \begin{bmatrix} y(2n) & y(2n-1)\ldots y(1) \\ \vdots & \vdots \qquad \vdots \\ y(N-1) & y(N-2) \ldots y(N-2n) \end{bmatrix},$$

$$V_e(N)=V_\varepsilon(N)+M_\varepsilon(N)A,\quad V_\varepsilon^T(N)=[\varepsilon(2n+1)\varepsilon(2n+2)\dots\varepsilon(N)],$$

$$M_\varepsilon(N)=\begin{bmatrix}\varepsilon(2n) & \varepsilon(2n-1) & \dots & \varepsilon(1)\\ \varepsilon(N-1) & \varepsilon(N-2) & \dots & \varepsilon(N-2n)\end{bmatrix}.$$

The symmetry conditions may be used to derive a symmetric model

$$qV_y(N) = -M_y(N)QA^*+V_e(N) \tag{2.3}$$

where $q=1+z^{-2n}$, while Q is a 2nxn matrix

$$Q^T=\begin{bmatrix}00 & \dots & 010 & \dots & 000\\ 00 & \dots & 101 & \dots & 000\\ : & & & & \\ 01 & \dots & 000 & \dots & 100\\ 10 & \dots & 000 & \dots & 010\end{bmatrix}$$

Starting from the signal model (2.1) different estimation algorithms can be defined. It is well known that the least squares (LS) algorithm

$$\hat{A}(N)=-[M_y^T(N)M_y(N)]^{-1}M_y^T(N)V_y(N) \tag{2.4}$$

gives asymptotically biased estimates $\hat{A}^T(N)=[\hat{a}_1(N)\ \hat{a}_2(N)\dots\hat{a}_{2n}(N)]$ of the parameters $A^T=[a_1a_2\dots a_{2n}]$ due to the correlation between signal samples $y(i)=x(i)+\varepsilon(i)$ and the noise term in the signal model (2.1) $e(i)=A(z^{-1})\varepsilon(i)$. An order to study the asymptotic behaviour of the algorithm the probability limit of the estimate

$$PL[\hat{A}(N)] = \operatorname*{plim}_{N\to\infty} \hat{A}(N) \tag{2.5}$$

and the asymptotic variance

$$AV[\hat{A}(N)] = \frac{1}{N}\operatorname*{plim}_{N\to\infty}\{N[\hat{A}(N)-PL[\hat{A}(N)]][\hat{A}(N)-PL[\hat{A}(N)]]^T\} \tag{2.6}$$

are calculated |3|. Inserting (2.2) in (2.4) we obtain

$$\hat{A}(N) = A-[M_y^T(N)M_y(N)]^{-1}M_y^T(N)V_e(N) \tag{2.7}$$

Taking probability limit and making use of a corollary of Slutsky's theorem |3,7| we obtain the following expression valid for algorithms of LS type

$$PL_{LS}[\hat{A}(N)] = A - R_y^{-1}P_e \tag{2.8}$$

where $R_y = \operatorname*{plim}_{N\to\infty}[\frac{1}{N} M_y^T(N)M_y(N)] = R_x + R_\varepsilon$, $[R_x]_{ij} = \sum_{m=1}^{n} \frac{1}{2} A_m^2\cos(i-j)\omega_m$, $[R_\varepsilon]_{ij} = r_\varepsilon(i-j)$ and $P_e = P_\varepsilon + R_\varepsilon A$, $P_\varepsilon = \operatorname*{plim}_{N\to\infty}[\frac{1}{N} M_\varepsilon^T(N)V_\varepsilon(N)] = [r_\varepsilon(1)r_\varepsilon(2)\dots\dots r_\varepsilon(2n)]^T$.

Similarly, starting from (2.6) and (2.7) we get the following expression for AV of LS type algorithms

$$AV_{LS}[\hat{A}(N)] = \frac{1}{N} R_y^{-1}\operatorname*{plim}_{N\to\infty}\{\frac{1}{N} M_\varepsilon^T(N)V_e(N)V_e^T(N)M_\varepsilon(N) - M_\varepsilon^T(N)V_e(N)P_e^T-P_eV_e^T(N)M_\varepsilon(N) + NP_eP_e^T\}R_y^{-T} \tag{2.9}$$

The LS algorithm may be applied to the symmetric model (2.3) yielding

$$\hat{A}^*(N)=-[Q^TM_y^T(N)M_y(N)Q]^{-1}Q^TM_y^T(N)qV_y(N) \tag{2.10}$$

The expressions for the corresponding PL and AV are

$$PL_{LS}[\hat{A}^*(N)] = A^* - R_y^{*-1}P_e^* \tag{2.11}$$

$$AV_{LS}[\hat{A}^*(N)] = \frac{1}{N} R_y^{*-1} \operatorname*{plim}_{N\to\infty}\{\frac{1}{N} Q^TM_\varepsilon^T(N)V_e^T(N)V_e^T(N)M_\varepsilon(N)Q - Q^TM_\varepsilon^T(N)V_e(N)P_e^{*T}-P_e^*V_e^T(N)M_\varepsilon(N)Q+NP_e^*P_e^{*T}\}R_y^{*-T} \tag{2.12}$$

where $R_y^*=Q^TR_yQ$ and $P_e^*=2Q^TP_\varepsilon+Q^TR_\varepsilon QA^*$.

In the case of white additive noise $\varepsilon(i)=\xi(i)$ PL of the LS estimate becomes

$$PL_{LS}[\hat{A}(N)]= A-(R_x+I\sigma_\xi^2)^{-1}\sigma_\xi^2A \tag{2.13}$$

$$PL_{LS}[\hat{A}^*(N)] = A^*-(Q^TR_xQ+Q^TQ\sigma_\xi^2)^{-1}Q^TQ\sigma_\xi^2A^* \tag{2.14}$$

As the signal to noise ratio ($SNR=10\log(\sum_{j=1}^{n} A_j^2/2\sigma_\xi^2)$) decreases, the asymptotic bias corresponding to the second term of (2.13) or (2.14) tends to $-A$ or $-A^*$, and estimates $\hat{A}(N)$ or $\hat{A}^*(N)$ tend to zero. For extremely low SNR the estimated polynomial $\hat{A}_N^*(z^{-1})=1+\hat{a}_1^*(N)z^{-1}+\dots+\hat{a}_1^*(N)z^{-2n+1}+z^{-2n}$ becomes almost equal to $1+z^{-2n}$, so that its roots are spread at equiangular distances on the unit circle in the z-plane. Particularly, if a second order symmetric model is used to estimate a single sinusoid's frequency ($Q^T=[1\ 0]$), the true value of the parameter a_1 is $a_1=2\cos\omega_1$, while

$$PL_{LS}[\hat{a}_1^*(N)] = \left[1 - \frac{1}{1+\frac{A_1^2}{2\sigma_\xi^2}}\right]a_1 \tag{2.15}$$

According to the expression (2.12) AV of the LS estimate for the second order symmetric model is

$$AV_{LS}[\hat{a}_1^*(N)] = \frac{1}{N}\,\frac{4(1+2\cos^2\omega_1)}{\left(\frac{A_1^2}{2\sigma_\xi^2}+1\right)^2} \tag{2.16}$$

GENERALIZED LEAST-SQUARES APPROACH

The basic idea of the GLS method is to prefilter the signal $y(i)$ in order to whiten the equivalent noise in the signal model (2.1) and, thus, to decorrelate the filtered signal and noise samples |7,8,4|. Obviously filtering

$$\tilde{y}(i) = y(i)/A(z^{-1}) \tag{3.1}$$

has this property provided additive noise is white. Having in mind that $1/A(z^{-1})$ is the matched filter for the signal $y(i)$ composed of sinusoids and additive white noise, it is clear that the output $\tilde{y}(i)$ has deterministic components with linearly increasing amplitudes at the original signal frequencies so that the effective signal to noise ratio is improved. However, the filter required to whiten the noise term in the signal model is not convenient since it is on the stability boundary. Contracting the filter poles in the z-plane towards the origin, a stable filter is obtained. As the result, the asymptotic bias cannot be eliminated, but it can be significantly reduced with respect to (2.11). The detailed analysis will be presented for the single sinusoid case.

The asymptotic estimate $PL_{GLS}[\hat{a}_1^*(N)]=\operatorname*{plim}_{N\to\infty} \hat{a}_1^*(N)$ is derived starting from the assumptions that the signal samples are filtered by the stabilized filter of center frequency $\arccos\{\frac{1}{2} PL_{GLS}[\hat{a}_1^*(N)]\}$

$$\frac{1}{\hat{A}_\infty^*(\alpha z^{-1})} = \frac{1}{1+ PL_{GLS}[\hat{a}_1^*(N)]z^{-1}+\alpha^2z^{-2}},\quad (0<\alpha<1) \tag{3.2}$$

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and that a LS-type algorithm is used in estimating the parameters of the model

$$qV_{\tilde{y}}(N) = -M_{\tilde{y}}(N)QA^{*}+V_{\tilde{e}}(N) \tag{3.3}$$

thus, the estimate is

$$\hat{A}^{*}(N) = -\left[Q^{T}M_{\tilde{y}}^{T}(N)M_{\tilde{y}}(N)Q\right]^{-1}Q^{T}M_{\tilde{y}}^{T}(N)qV_{\tilde{y}}(N) \tag{3.4}$$

Here $M_{\tilde{y}}(N)$, $V_{\tilde{y}}(N)$, $V_{\tilde{e}}(N)$ represent filtered versious of $M_y(N)$, $V_y(N)$ and $V_e(N)$, respectively. Accordingly, filtered versions of $M_\xi(N)$ and $V_\xi(N)$ will be denoted by $M_{\tilde{\xi}}(N)$ and $V_{\tilde{\xi}}(N)$, while $\tilde{y}(i) = y(i)/A_\infty^{*}(\alpha z^{-1})$, $\tilde{x}(i) = x(i)/A_\infty^{*}(\alpha z^{-1})$, $\tilde{\xi}(i) = \xi(i)/A_\infty^{*}(\alpha z^{-1})$ and $\tilde{e}(i) = e(i)/A_\infty^{*}(\alpha z^{-1})$ denote filtered values of $y(i)$, $x(i)$, $\xi(i)$ and $e(i)$, respectively. The amplitude $\tilde{A}_1$ of the sinusoid $\tilde{x}(i)$ is easily determined by substituting $e^{j\omega_1}$ for z in (3.2). Introducing

$f = \dfrac{PL_{GLS}[\hat{a}_1^{*}(N)]}{a_1}$ it follows that the amplification $g = \tilde{A}_1/A_1$ achieved by filtering is equal to $g = |1+2\alpha f e^{-j\omega_1}\cos\omega_1+\alpha^2 e^{-j2\omega_1}|^{-2}$, which yields

$$g = \frac{1}{(1-\alpha^2)^2+\alpha(\alpha-f)(1-\alpha f)a_1^2} \tag{3.5}$$

The filtered noise can be represented as

$$\tilde{\xi}(i) = \sum_{j=o}^{\infty} k_j\xi(i-j) \tag{3.6}$$

where

$k_j = 0$ for $j<0$, $k_o=1$, $k_j=-\alpha f a_1 k_{j-1}-\alpha^2 k_{j-2}$ for $j>0$.

These coefficients are easily found to have the following properties

$$\lim_{j\to\infty} k_j = 0 \tag{3.7}$$

$$K_o = \sum_{j=o}^{\infty} k_j^2 = \frac{1+\alpha^2}{1-\alpha^2}\,\frac{1}{(1+\alpha^2)^2-(\alpha f a_1)^2}$$

$$K_1 = \sum_{j=o}^{\infty} k_j k_{j+1} = \frac{-\alpha f a_1}{1+\alpha^2}\,K_o$$

Proceeding as in the case of ordinary LS algorithm, studied in section 2, we obtain

$$PL_{GLS}[\hat{A}^{*}(N)] = A^{*} - R_{\tilde{y}}^{*-1}P_{\tilde{e}}^{*} \tag{3.8}$$

where $R_{\tilde{y}}^{*} = Q^{T}R_{\tilde{y}}Q$, $P_{\tilde{e}}^{*} = 2Q^{T}P_{\tilde{\xi}}+Q^{T}R_{\tilde{\xi}}QA^{*}$, $R_{\tilde{y}} = \operatorname{plim}_{N\to\infty}[\frac{1}{N}M_{\tilde{y}}^{T}(N)M_{\tilde{y}}(N)]=R_{\tilde{x}}+R_{\tilde{\xi}}$, $[R_{\tilde{x}}]_{ij} = \sum_{m=1}^{n}\frac{1}{2}A_m^2\cos(i-j)\omega_m$, $[R_{\tilde{\xi}}]_{ij} = r_{\tilde{\xi}}(i-j)$ and $P_{\tilde{\xi}} = \operatorname{plim}_{N\to\infty}[\frac{1}{N}M_{\tilde{\xi}}^{T}(N)V_{\tilde{\xi}}(N)] = [r_{\tilde{\xi}}(1)\; r_{\tilde{\xi}}(2)\; \dots\; r_{\tilde{\xi}}(2n)]^{T}$. In the one dimensional case

$$PL_{GLS}[\hat{a}_1^{*}(N)] = a_1 - \frac{2r_{\tilde{\xi}}(1)+r_{\tilde{\xi}}(0)a_1}{r_{\tilde{x}}(0)+r_{\tilde{\xi}}(0)} \tag{3.9}$$

which is readily represented as

$$PL_{GLS}[a_1^{*}(N)] = a_1 - \frac{2K_1\sigma_\xi^2 + K_o\sigma_\xi^2 a_1}{gA_1^2/2 + K_o\sigma_\xi^2} \tag{3.10}$$

Owing to (3.7)

$$f = 1 - \frac{(1-\alpha)^2+2\alpha(1-f)}{(1+\alpha^2)(1+\frac{g}{K_o}\,\frac{A_1^2}{2\sigma_\xi^2})} \tag{3.11}$$

Introducing

$$G = \frac{1+\alpha}{(1-\alpha)^2}^{2}\,\frac{g}{K_o} \tag{3.12}$$

and rearanging equation (3.11) we obtain

$$f = 1 - \frac{1}{1+G\,\frac{A_1^2}{2\sigma_\xi^2}} \tag{3.13}$$

Comparing (3.13) with (2.15), previously derived for the LS estimator, it can be seen that G plays the role of the effective, over-all gain due to filtering. Substituting for g and K_o from (3.5) and (3.7) we obtain

$$G = \frac{1+\alpha}{(1-\alpha)^3}\,\frac{(1+\alpha^2)^2-\alpha^2f^2a_1^2}{(1+\alpha)^2-\alpha\frac{(f-\alpha)(1-\alpha f)}{(1-\alpha)^2}a_1^2} \tag{3.14}$$

The pair of equations (3.13) and (3.14) defines the asymptotic value of the GLS estimate.

For $1-\alpha\ll1$, $1-\alpha\ll\frac{A_1^2}{2\sigma_\xi^2}$ and $(1-\alpha)^2\ll1-\cos\omega_1$ one obtains, approximately,

$$f = 1 - \frac{(1-\alpha)^3\sigma_\xi^2}{A_1^2} \tag{3.15}$$

$$G = \frac{2}{(1-\alpha)^3}$$

An approximate expression for $AV_{GLS}|a_1^{*}(N)|$ will be derived starting from the assumption that signal samples are filtered by the stable filter (3.2). However, for close to unity the asymptotic bias may be neglected according to (3.15), and thus the center frequency of the filter may be considered equal to the algorithm (3.4) is that of the LS, $AV_{GLS}[a_1^{*}(N)]$ is derived on the basis of (2.12) by substituting R_y^{*}, P_e^{*}, M and V_e by their filtered versions, which correspond to the second order symmetric model of a single sinusoid in white additive noise: $R_{\tilde{y}}^{*}$, $P_{\tilde{e}}^{*}$, M and V_e, respectively. In this way, we obtain

$$AV_{GLS}[\hat{a}_1^{*}(N)] = \frac{1}{N}\,\frac{\operatorname{plim}_{N\to\infty}\{\frac{1}{N}s^2(N)-2Ss(N)+NS^2\}}{\left[\operatorname{plim}_{N\to\infty}\{\frac{1}{N}\sum_{j=3}^{N}[\sum_{m=o}^{\infty}k_m\xi(j-m-1)]^2\}\right]^2} \tag{3.16}$$

where

$$s(N) = \sum_{j=3}^{N}\{\sum_{m=o}^{\infty}[k_m\xi(j-m-1)]\sum_{m=o}^{\infty}[(k_m+a_1k_{m-1}+k_{m-2})\xi(j-m)]\}$$

$$S = \operatorname{plim}_{N\to\infty}\{\frac{1}{N}s(N)\}$$

After a straightforward but rather tedious derivation one obtains for $\alpha\simeq1$

$$AV_{GLS}[a_1^{*}(N)] = \frac{2}{N}\,\frac{(1-\alpha)^3(1-\cos^4\omega_1)}{\left(\frac{A_1^2}{2\sigma_\xi^2}\right)^2} \tag{3.17}$$

Notice that the obtained error variance offers an improvement when compared to the $AV_{LS}[a_1^{*}(N)]$ given in (2.15) and $AV_{LS}[a_1^{*}(N)]$ derived in |3| for the instrumental variable (IV) method.

FREQUENCY ESTIMATION ALGORITHM

The application of the GLS methodology to sinusoidal signals requires no special steps to estimate the filter parameters |7,8|. The algorithm makes use of the previous estimates of parameters A, i.e. in the k-th iteration the signal samples are filtered by the filter $1/\hat{A}^{*(k-1)}_N(\alpha_k z^{-1})$, $0<\alpha_k<1$ based on the parameter vector estimate $\hat{A}^{*(k-1)T}(N) = [\hat{a}^*_1(N)\ \hat{a}^*_2(N) \dots \hat{a}^*_n(N)]$ obtained in the preceeding iteration, so that

$$\hat{A}^{*(k-1)}_N(\alpha_k z^{-1}) = 1+\hat{a}^{*(k-1)}_1(N)\alpha_k z^{-1}+\hat{a}^{*(k-1)}_2(N)\alpha_k^2 z^{-2}+ +\dots+\hat{a}^{*(k-1)}_1(N)\alpha_k^{2n-1}z^{-2n+1}+\alpha_k^{2n}z^{-2n}$$ and

$$\tilde{y}^{(k)}(i) = \frac{y(i)}{\hat{A}^{*(k-1)}_N(\alpha_k z^{-1})} \qquad (4.1)$$

Then, the estimate $\hat{A}^{*(k)}(N)$ is obtained by applying the LS algorithm to the filtered signal samples $y^{(k-1)}(i)$:

$$\hat{A}^{*(k)}(N) = -[Q^T M^{(k-1)T}_{\tilde{y}}(N) M^{(k-1)}_{\tilde{y}}(N) Q]^{-1} Q^T M^{(k-1)T}_{\tilde{y}}(N) \cdot \cdot qV^{(k-1)}_{\tilde{y}}(N) \qquad (4.2)$$

In the first iteration the estimate $A^{*(1)}(N)$ is obtained on the basis of the original signal sample sequence y(i). For low SNR this estimate may be highly incorrect, so that taking $\alpha\approx 1$, one may inhance false frecuencies. As k increases and $N\to\infty$, $A^{*(k)}(N)$ should tend to the value $PL_{GLS}[\hat{A}^*(N)]$ which is, according to the results of the previous section closer to the true value A for α_k closer to one. Therefore, the choice of the sequence α_k is of an extreme importance. It depends highly upon the uncertainely about the true value. The closer the initial LS estimate of $A(z^{-1})$ is to $1+z^{-2n}$, the higher this uncertainely is.

For a single sinusoid the greatest disagreement between the $PL_{LS}[a^*(N)]$ and the value a_1 occurs in the cases of very low sinusoid frequencies and very low SNR. To derive the optimal sequence for this case we proceed as follows.

The estimate $\hat{a}^{*(k)}_1(N)$ obtained in the k-th iteration on the basis of signal samples filtered by $1/\hat{A}^{*(k-1)}_N(\alpha_k z^{-1}) = 1/(1+\alpha_k \hat{a}^{*(k-1)}_1(N)z^{-1}+\alpha_k^2)$ is, in analogy with (3.11),

$$\hat{a}^{*(k)}_1(N) = a_1\left[1 - \frac{(1-\alpha_k)^2+2\alpha_k\left(1-\frac{\hat{a}^{*(k-1)}_1(N)}{a_1}\right)}{(1+\alpha_k^2)\left(1+\frac{g^{(k-1)}}{K_o^{(k-1)}}\frac{A_1^2}{2\sigma_\xi^2}\right)}\right] \qquad (4.3)$$

where

$$\frac{g^{(k-1)}}{K_o^{(k-1)}} = \frac{1-\alpha_k^2}{1+\alpha_k^2}\ \frac{(1+\alpha_k^2)^2-\alpha_k^2\hat{a}^{*(k-1)2}_1(N)}{(1-\alpha_k^2)^2+\alpha_k(\alpha_k a_1-\hat{a}^{*(k-1)}_1(N))(a_1-\alpha_k\hat{a}^{*(k-1)}_1(N))} \qquad (4.4)$$

(4.3) may be written in the form

$$\hat{a}^{*(k)}_1(N) = a_1\left[1 - \frac{1}{1+G^{(k)}\frac{A_1^2}{2\sigma_\xi^2}}\right] \qquad (4.5)$$

where

$$G^{(k)} = \frac{2\alpha_k G^{(k-1)}+(1+\alpha_k^2)(1+G^{(k-1)}\frac{A_1^2}{2\sigma_\xi^2})\frac{g^{(k-1)}}{K_o^{(k-1)}}}{1+\alpha_k^2+(1-\alpha_k)^2 G^{(k-1)}\frac{A_1^2}{2\sigma_\xi^2}} \qquad (4.6)$$

For the worst case of zero frequency and maximum bias the ratio $g^{(k-1)}/K_o^{(k-1)}$ becomes

$$\frac{g^{(k-1)}}{K_o^{(k-1)}} = \frac{1-\alpha_k^2}{1+\alpha_k^2} \qquad (4.7)$$

Inserting (4.7) into (4.6) we obtain

$$G^{(k)} = \frac{1+2\alpha_k G^{(k-1)}-\alpha_k^2+(1-\alpha_k^2)G^{(k-1)}(A_1^2)/(2\sigma_\xi^2)}{1+\alpha_k^2+(1-\alpha_k)^2 G^{(k-1)}(A_1^2)/(2\sigma_\xi^2)} \qquad (4.8)$$

Looking for maxima of expression (4.8) for k=1,2,3,... and neglecting terms with $(A_1^2)/(2\sigma_\xi^2)$ we obtain

$$\alpha_k = (\sqrt{k}-1)/\sqrt{k-1} \qquad G^{(k)} = \sqrt{k} \qquad (4.9)$$

This is the rate of improvement in SNR for the worst case. Assuming less severe conditions different sequencies of α_k can be established to yield faster convergence. For example for a high SNR the choice of constant α_k close to unity may be suitable.

The same methodology may be applied to the nonasymmetric signal model, as well.

COLORED NOISE CASE

GLS method can easily be extended in order to deal with the problem of sinusoids in additive MA noise $\varepsilon(i)=B(z^{-1})\xi(i)$. The extended GLS (EGLS) algorithm designed to estimate both A and B parameters of the model

$$A(z^{-1})y(i) = A(z^{-1})B(z^{-1})\xi(i) \qquad (5.1)$$

has the following form

$$\hat{\theta}^{*(k)}(N) = -\left[M^{(k-1)T}_{\tilde{y}\hat{n}}(N) M^{(k-1)}_{\tilde{y}\hat{n}}(N)\right]^{-1} M^{(k-1)T}_{\tilde{y}\hat{n}}(N) qV^{(k-1)}_{\tilde{y}}(N) \qquad (5.2)$$

$$\hat{n}^{(k)}(i) = \hat{A}^{(k)}_N(z^{-1})\tilde{y}^{(k-1)}(i) - [\hat{B}^{(k)}_N(\alpha_k z^{-1})-1]\hat{n}^{(k-1)}(i) \qquad (5.3)$$

$$\tilde{y}^{(k)}(i) = y(i)/\hat{A}^{(k)}_N(\alpha_k z^{-1}) \qquad (5.4)$$

where

$$\theta^{*(k)T}(N) = [A^{*(k)T}(N) B^{(k)T}(N)] \qquad (5.5)$$

is the extended vector of parameter estimates in k-th iteration and

$$M^{(k)}_{\tilde{y}\hat{n}}(N) = [M^{(k)}_{\tilde{y}}(N)Q \quad -M^{(k)}_{\hat{n}}(N)] \qquad (5.6)$$

$M^{(k)}_{\hat{n}}(N)$ being a matrix formed of elements $y^{(k)}(i)$, i=N-1, N-2, ... arranged in the usual way.

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A nonsymmetric version of the algorihtm can readily be written.

One can immediately notice the resemblance to extended least squares (ELS) and maximum likelihood (ML) methodsbased on the signal model $A(z^{-1})y(i)=C(z^{-1})\xi(i)$ |7,5|. However, the proposed EGLS method makes use of the prior knowledge that the system noise term is of the form $e(i)=A(z^{-1})B(z^{-1})\xi(i)$. In this way it avoids redundency in the number of parameters and requires lower model orders than ELS or ML method for the same signal. In addition, the EGLS method provides directly the noise parameters estimates.

EXPERIMENTAL RESULTS

An extensive experimental analysis of the algorithm has been undertaken. Some illustrative computer simulation results are shown bellow.

The debiasing effect is demonstrated for a single sinusoid of constant frequency $\omega_1=0.4\pi$ uniformly sampled in 1000 points and for three values of SNR. The assumed model was second order symmetric and $\alpha=.9$. Table 1 presents frequency estimate mean values and standard deviations obtained both analyticaly (PL and AV) and by means of Monte Carlo simulations based on 30 noisy signal realizacions (EMV and ESD). A high agreement between theoretical and experimental values is evident, as well as the reduction of the bias and variance, compared to the LS case.

Figure 1 represents typical convergence behaviour of the iterative GLS algorithm for four values of SNR. 1000 samples of a single noisy sinusoid were used. The signal model was second order symmetric and $\alpha=9$. Frequency estimates after each iteration are given, showing a faster convergence for higher SNR values. In the first iteration the estimates were obtained applying the LS algorithm.

The application of EGLS algorithm to the case of multiple sinusoids and additive colored noise is illustrated in figure 2, and compared to the ML and IV algorithms, the signal was composed of four sinusoids with frequencies $\omega_1=0.35\pi$, $\omega_2=0.4\pi$, $\omega_3=0.7\pi$ and $\omega_4=0.7\pi$ and additive second order MA noise with $B^T=[0.5\ \ -0.2]$ and SNR=0 dB. 2000 signal samples were used and signal models were nonsymmetric. The model order was 2n=8 and p=2 for EGLS, while p=10 for ML. The EGLS algorithm obviously has a better resolution and accuracy.

CONCLUSION

In this paper an analysis of the application of the generalized least-squares method to the estimation of frequencies of sinusoids in additive colored noise is given. A detailed theoretical analysis of the asymptotic properties of the estimates is presented. The derived expressions for the asymptotic bias and error covariance represent the basis for a qualitative analysis of the advantages and disadvantages the method, as well as for the derivation of an algorithm able to solve the posed problem efficiently in practice. From this point of view, the analysis related to the way of choosing the factor contracting the poles of data filter is especially important. The proposed algorithm provides good results even in the case of low SNR ratios. It possesses certain advantages over frequently applied versions of the maximum likelihood and instrumental variable methods.

Further investigations could be oriented towards making the contraction factor data dependent. This would allow a high adaptivity of the method in the case of imprecise a priori knowledge.

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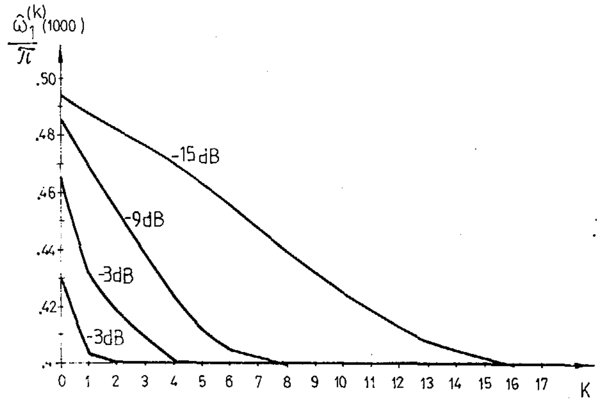

Figure 1.

SNR [dB]	PL_{LS}	EMV_{LS}	PL_{GLS}	EMV_{GLS}	$AV_{LS}^{\frac{1}{2}}$	ESD_{LS}	$AV_{GLS}^{\frac{1}{2}}$	ESD_{GLS}
3	0.43396	0.43550	0.40003	0.40005	0.00385	0.00422	0.00012	0.00023
-3	0.46715	0.46830	0.40048	0.40048	0.00770	0.00781	0.00047	0.00066
-9	0.48901	0.48898	0.40012	0.40013	0.01027	0.00946	0.00189	0.00236

TABLE 1

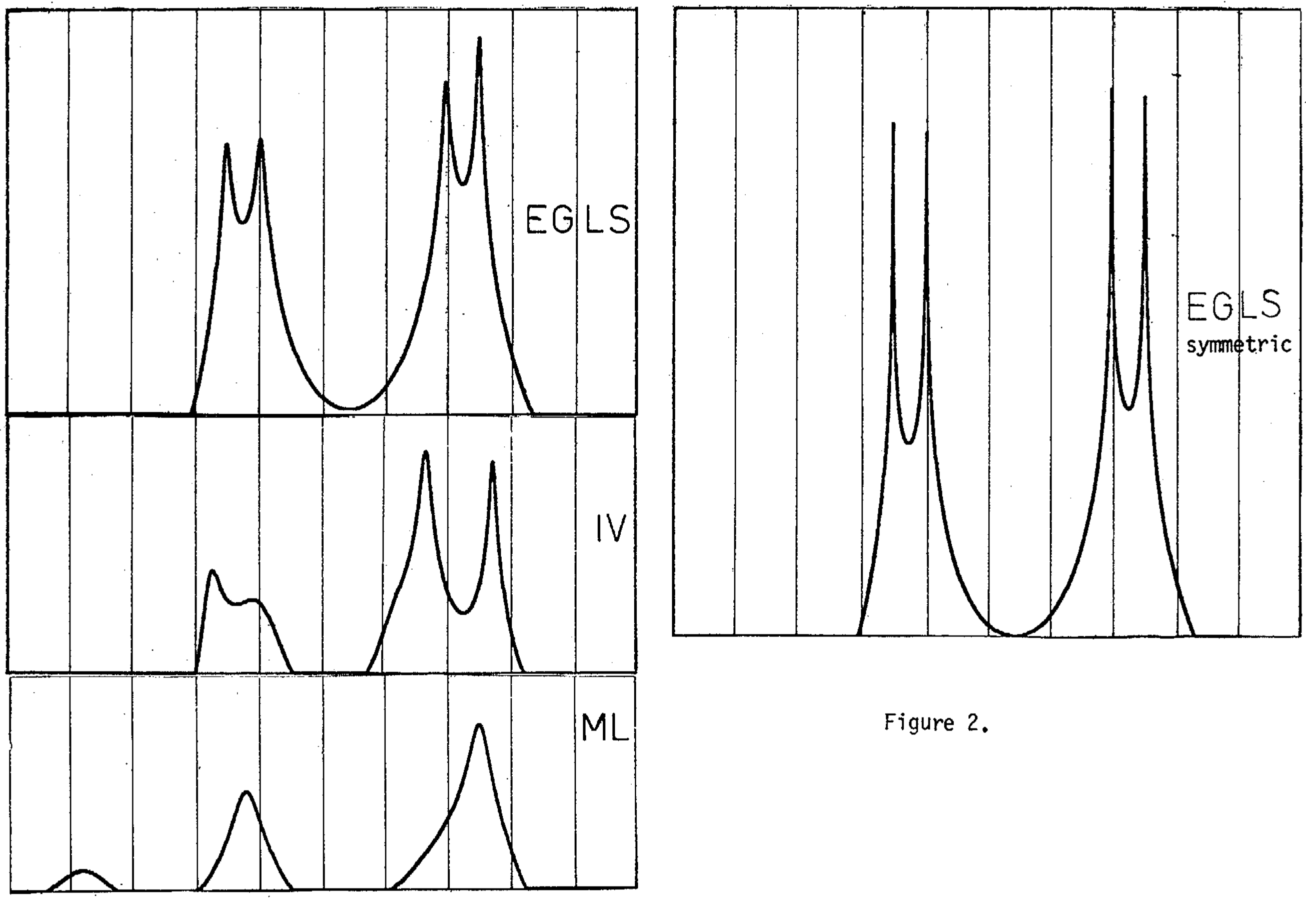

Figure 2.