DIXIEME COLLOQUE SUR LE TRAITEMENT DU SIGNAL ET SES APPLICATIONS

NICE du 20 au 24 MAI 1985

A THEORETICAL AND EXPERIMENTAL INVESTIGATION FOR SPECTRAL ESTIMATION OF HARD LIMITED OBSERVATIONS

D.F. Gingras

Naval Ocean Systems Center, San Diego, CA 92152-5000 USA

RESUME

Dans nombre d'applications, on peut désirer fortement contraindre (un seul niveau de discrétisation en amplitude) la mesure d'une série temporelle avant toute transmission numérique et traitement ultérieur. Lorsque l'on s'intéresse uniquement à l'examen de la structure spectrale de la série observée après transmission et non pas à la reconstruction de la série originelle, on a montré (1), (2) qu'il existe des méthodes d'estimation cohérentes. Cet article considère considère les résultats obtenus par les estimés de la densité spectrale de taille d'echantillonnage finie pour un processus temporel discret gaussien stationnaire à partir des observations d'une version fortement contrainte.

On utilise comme estimateur de la densité spectrale celui proposé par Rodemich (3) pour les fonctions non-linéaires des processus stochastiques gaussiens. Cet estimateur diffère du lissage habituel du périodogramme en ce que l'on applique une fonction non-linéaire aux estimés de la covariance avant de calculer l'estimé du spectre. Dans le cas fortement contraint, la fonction non-linéaire est la fonction sinus.

Dans cet article, nous présentons brièvement les résultats statistiques asymptotiques de (2) sous la forme d'expressions pour la moyenne et la variance asymptotiques et nous démontrons que l'estimateur est cohérent au sens de la moyenne quadratique. Nous présentons aussi les conclusions d'une analyse empirique des résultats pour une taille d'échantillonnage finie.

L'analyse empirique considère deux exemples, l'un avec une densité spectrale à "bande étroite", l'autre avec une densité spectrale à "bande large". Dans les deux cas, on a engendré les échantillons de séries temporelles à l'aide d'un modèle autorégressif à processus d'innovation gaussien. On a ensuite fortement contraint ces échantillons et calculé des estimés de la covariance. On a enfin calculé des estimés du spectre de taille d'échantillonnage finie pour la version fortement contrainte et pour la version fortement contrainte avec application de la fonction sinus aux estimés de la covariance. On montre la supériorité de la deuxième méthode sur un domaine de variation de la taille d'échantillonnage en évaluant une erreur quadratique moyenne en fréquence normalisée entre la densité spectrale connue exactement et les estimés du spectre.

SUMMARY

In a number of applications it may be desirable to hard limit (one level of amplitude quantization) an observed time series prior to digital transmission and subsequent processing. If one is only interested in examining the spectral structure of the observed series after transmission and not in reconstruction of the original series then consistent estimation methods have been shown to exist [1], [2]. In this paper the performance of finite sample size spectral density estimates of a stationary discrete-time Gaussian process from observations of a hard limited version is considered.

The spectral density estimator used is that proposed by Rodemich [3] for nonlinear functions of Gaussian random processes. This estimator differs from the usual smoothed periodogram in that a nonlinear function is applied to the covariance estimates prior to computing the spectral estimate. For the hard limited case this nonlinear function is the sine function.

In this paper we briefly present the asymptotic statistical results of [2] in the form of expressions for the asymptotic mean and variance and proof that the estimate is consistent in the quadratic-mean sense. We also present the results of an empirical investigation of finite sample size performance.

The empirical investigation considers two examples, one with a "narrowband" spectral density and one with a "broadband" spectral density. For both examples time series samples were generated using an autoregressive model with Gaussian innovations process. These samples were hard limited and covariance estimates were computed. Finite sample size spectral estimates were computed for the hard limited version and for the hard limited version with the sine function applied to the covariance estimates. The superiority of the latter is demonstrated for a range of sample sizes by evaluation of a normalized frequency averaged squared error between the known exact spectral density and the spectral estimates.

A THEORETICAL AND EXPERIMENTAL INVESTIGATION FOR SPECTRAL ESTIMATION OF HARD LIMITED OBSERVATIONS

D.F. Gingras

I. INTRODUCTION

We consider the problem of estimating the spectral density of a discrete-time stationary Gaussian process from observations of a hard limited version. In many applications it may be necessary to hard limit (i.e., one level of amplitude quantization) an observed random time series prior to digital transmission and subsequent processing. In some situations one may not be interested in reconstruction of the original time series from the hard limited version but only in estimating its spectral density. The accuracy to which this spectral estimate can be computed from the hard limited observations is empirically investigated. We examine both a narrow-band and a broadband spectral density example.

It has previously been shown, see Brillinger [1] and Gingras and Masry [2], that through the use of a nonlinear transformation applied to the estimated covariances of the hard limited process an asymptotically mean-square consistent estimate of the original spectral density can be constructed. In this paper we examine finite sample size spectral estimates formed directly from the covariance estimates of the hard limited series and formed using the nonlinearly transformed covariance estimates. These estimated spectra are compared with the exact spectral densities. We also compute a normalized frequency averaged square error as a function of the number of observations.

As indicated previously, a one-to-one transformation relating the covariance sequence $\{c_k\}$ of the output of a zero-memory nonlinearity to the covariance sequence $\{r_k\}$ at the input has been shown to exist for a class of nonlinearities, including the hard limiter. For the hard limiter case we have

$$r_k = \sin\{(\pi/2)c_k\} \quad . \tag{1}$$

Thus, an estimate for r_k based on N observations of the output is

$$\hat{r}_{N,k} = \sin\{(\pi/2)\hat{c}_{N,k}\} \tag{2}$$

where $\hat{c}_{N,k}$ is the usual biased estimate of the output covariance c_k. This leads to an estimate $\hat{\phi}_N(\lambda)$ of the input spectral density $\phi(\lambda)$ of the form

$$\hat{\phi}_N(\lambda) = (1/2\pi)\left\{1+2\sum_{k=1}^{N-1} h_N(k)\sin\left[\left(\frac{\pi}{2}\right)\hat{c}_{N,k}\right]\cos(k\lambda)\right\} \tag{3}$$

where $h_N(k)$ is a covariance averaging sequence to be specified below.

In Section II we briefly outline the asymptotic bias and covariance properties of the spectral estimate (3). In Section III we compare the performance of (3) with that of a smoothed periodogram, via Monte Carlo simulation, for two spectral density examples. In particular we compare both estimates to known exact spectral densities.

II. THEORETICAL RESULTS

We assume that the "input" sequence $X = \{X_k\}_{k=-\infty}^{\infty}$ is a real stationary Gaussian process with mean zero and absolutely summable covariance sequence $\{r_k\}$ (normalized with $r_0 = 1$). Then its spectral density $\phi(\lambda)$ exists and is given by

$$\phi(\lambda) = (1/2\pi)\sum_{k=-\infty}^{\infty} r_k e^{-ik\lambda} \quad . \tag{4}$$

Given a finite set of observations of the hard limited process Y, we obtain an estimate for the input spectral density $\phi(\lambda)$ as follows: first estimate the output covariance sequence $\{c_k\}$ by

$$\hat{c}_{N,k} = \begin{cases} \frac{1}{N}\sum_{j=1}^{N-|k|} Y_{|k|+j}Y_j, & |k| \le N-1 \\ 0 & |k| > N-1 \end{cases} \quad . \tag{5}$$

Then estimate $\phi(\lambda)$ by

$$\hat{\phi}_N(\lambda) = (1/2\pi)\left\{1+2\sum_{k=1}^{N-1} h_N(k)\sin\left[\left(\frac{\pi}{2}\right)\hat{c}_{N,k}\right]\cos(k\lambda)\right\} \tag{6}$$

where $\{h_N(k)\}$ is a covariance averaging sequence generated as follows:

$$h_N(k) = h(k/M_N) \ ; \ k = 1,2,\cdots$$

where M_N is a sequence of positive integers such that $M_N \to \infty$ and $M_N/N \to 0$ as $N \to \infty$; the function h(t) is real, even, h(0) = 1 and

$$|h(t)| \le \text{const.}/(1 + |t|)^{\frac{1}{2}+\varepsilon} \quad , \quad \varepsilon > 0 \quad . \tag{7}$$

For future reference we note that an estimate for the spectral density $\psi(\lambda)$ of the hard limited process Y is given by

$$\hat{\psi}_N(\lambda) = (1/2\pi)\left\{1 + 2\sum_{k=1}^{N-1} h_N(k)\hat{c}_{N,k}\cos(k\lambda)\right\} \quad . \tag{8}$$

The statistical properties of $\hat{\psi}_N(\lambda)$ are well known [4]. Those of the input spectral estimate $\hat{\phi}_N(\lambda)$ will be presented next. The proofs of the statistical results are contained in [2]. Similar results for continuous time processes are contained in [1].

The bias of the estimate (6) is given by the following theorem and its corollary.

<u>Theorem 1</u>. We have

$$E[\hat{\phi}_N(\lambda)] = (1/2\pi)\left\{1 + 2\sum_{k=1}^{N-1} h_N(k)r_k\cos(k\lambda)\right\} + O\left\{(M_N/N)^{\frac{1}{2}+\alpha(\varepsilon)}\right\} \tag{9}$$

where the $O(\cdot)$ term is uniform in λ and

$$\alpha(\varepsilon) = \begin{cases} \varepsilon, & \varepsilon < \frac{1}{2} \\ < \frac{1}{2}, & \varepsilon = \frac{1}{2} \\ \frac{1}{2}, & \varepsilon > \frac{1}{2} \end{cases} \tag{10}$$

A THEORETICAL AND EXPERIMENTAL INVESTIGATION FOR SPECTRAL ESTIMATION OF HARD LIMITED OBSERVATIONS

D.F. Gingras

Corollary. If, in addition

$$\sum_{k=-\infty}^{\infty} |k^p r_k| < \infty \qquad p = 1,2,\cdots,q$$

and h(t) is q times differentiable with bounded derivatives then

$$E[\hat{\phi}_N(\lambda)] = \phi(\lambda) + \sum_{p=1}^{q-1} \frac{(i)^p}{p!} \frac{h^{(p)}(0)}{M_N^{\,p}} \phi^{(p)}(\lambda)$$

$$+ O\{(1/M_N)^q\} + O\{(M_N/N)^{\frac{1}{2}+\alpha(\varepsilon)}\} \quad . \tag{11}$$

Next we present the covariance for the spectral estimate $\hat{\phi}_N(\lambda)$. In Theorem 2 below it is given in terms of the covariance of the output spectral estimate $\hat{\psi}_N(\lambda)$ of (8).

Theorem 2. We have

$$\mathrm{cov}[\hat{\phi}_N(\lambda),\hat{\phi}_N(\mu)] = (\pi/2)^2 \,\mathrm{cov}[\hat{\psi}_N(\lambda),\hat{\psi}_N(\mu)]$$

$$+ O\left\{(M_N/N)\left[(M_N/N)^{\alpha(\varepsilon)} + (1/M_N)^{\frac{1}{2}}\right]\right\} \tag{12}$$

the $O(\cdot)$ term is uniform in λ and μ and $\alpha(\varepsilon)$ is given by (10). Since the output spectral estimate $\hat{\psi}_N(\lambda)$ is a standard smoothed periodogram its asymptotic covariance is well-known (see for example, [4, Theorem 5A]).

Corollary. We have

$$\lim_{N\to\infty}(N/M_N)\mathrm{var}[\hat{\phi}_N(\lambda)] = [(\pi/2)\psi(\lambda)]^2 \int_{-\infty}^{\infty} h^2(t)dt(1+\bar{\delta}_{0,\lambda}) \tag{13}$$

where

$$\bar{\delta}_{0,\lambda} = \begin{cases} 1, & \text{for } \lambda = 0 \pmod{2\pi} \\ 0, & \text{otherwise} \end{cases} \quad .$$

We note that the asymptotic variance of the input spectral estimate $\hat{\phi}_N(\lambda)$ is identical to that of a smoothed periodogram except for the multiplicative factor $(\pi/2)^2$ introduced by hard limiting. The corollaries to Theorems 1 and 2 imply the mean-square consistency of the input spectral estimate $\hat{\phi}_N(\lambda)$ as $N\to\infty$.

Let ν^2 be the asymptotic normalized variance for the spectral estimate, using (13)

$$\nu^2 = \lim_{N\to\infty} (\pi/2)^2 \frac{\mathrm{var}\{\hat{\psi}_N(\lambda)\}}{\psi^2(\lambda)}$$

$$= (M_N/N)(\pi/2)^2 \int_{-\infty}^{\infty} h^2(t)dt \ . \tag{14}$$

For the Bartlett lag window, which was used in the empirical investigation, we have that

$$\int_{-\infty}^{\infty} h^2(t)dt = 2/3 \quad .$$

Fig. 1 illustrates the results of evaluating (14) for a range of values for M_N/N. In the next section we will compare these asymptotic results with those obtained empirically.

III. SIMULATION RESULTS

In this section X is a stationary Gaussian process with zero mean, summable covariance sequence $\{r_k\}$ and a spectral density function $\phi(\lambda)$; the process Y is a hardlimited version of X. By the results of [5], we are guaranteed that the covariance sequence $\{c_k\}$ for the Y process is summable and that the spectral density $\psi(\lambda)$ exists. We examine through simulation the performance of finite sample-size spectral estimates of the X process formed from observations of the Y process for two spectral density examples.

The example spectral densities used in the following empirical investigation are based on finite order autoregressive-moving average (ARMA) models. This spectral model was chosen because of the ease of generating time series data with known spectral density, and because most useful spectra can be modeled by a rational spectral density model. Two autoregressive (AR) examples representing different spectral characteristics were chosen. For the first example the AR parameters were chosen so as to produce a narrow band spectral density with a well defined peak. The actual spectral density is illustrated by Fig. 2(A). The second example was chosen to be a broadband example, that is, a smooth spectral density with a broad "hump" but no spectral peaks. See Fig 3(A) for the actual spectral density.

For the empirical investigation the random time series X was generated according to the following AR model

$$X_n = a_1X_{n-1} + a_2X_{n-2} + \cdots + a_pX_{n-p} + \varepsilon_n \tag{15}$$

where the AR parameters $\left\{a_j\right\}_{j=1}^{p}$ are specified below for the two examples considered and the noise sequence ε_n is a sequence of zero-mean i.i.d. Gaussian random numbers with variance σ_ε^2 obtained by applying the Box-Muller transformation to uniform [0,1] random numbers and scaling by the variance σ_ε^2. For both spectral examples ten realizations of the AR series were generated; the first few hundred samples were not used so as to eliminate transient effects. All series were subjected to hardlimiting and sequences of covariance estimates were computed using (5), for N = 1.024K, 5.12K, 10.24K, 20.48K, 40.96K, and normalized by $\hat{c}_{N,0}$. Spectral estimates were computed with the covariance transformation (6), and standard smoothed periodograms were computed. The covariance averaging function used for both estimates is referred to as the Bartlett lag window and is given by

$$h(k) = \begin{cases} 1 - |k|/M, & |k| \le M \\ 0 \quad , & |k| > M \end{cases} \tag{16}$$

where M is the usual "window parameter" that defines the truncation point of the covariance sequence and determines the "resolution" of the spectral estimate.

A THEORETICAL AND EXPERIMENTAL INVESTIGATION
FOR SPECTRAL ESTIMATION OF HARD LIMITED OBSERVATIONS

D.F. Gingras

The parameter M was chosen empirically to be large enough to provide adequate spectral resolution and small enough so that the spectral estimates were stable (i.e., small variance). The value used throughout the investigation was M = 128.

To aid in the empirical comparison of the spectral estimate (6) with the smoothed periodogram, we estimated a normalized and frequency averaged error. The estimate is defined as

$$v^2 = \frac{1}{M}\sum_{k=0}^{M-1} \frac{[\hat{\phi}_N(\lambda_k) - \phi(\lambda_k)]^2}{\phi^2(\lambda_k)} \qquad (17)$$

In the results presented below, v^2 was computed for each of the ten realizations of the AR series and the values presented, $\bar{v}^2$, are an average over all ten realizations.

A. The AR (4) Example - Narrowband Spectral Density

The parameters chosen for this example are:

$p = 4 \quad a_1 = 1.60 \quad a_2 = -1.30 \quad a_3 = 0.80$

$a_4 = -0.40 \quad \sigma_\varepsilon^2 = 1.0.$

The actual spectral density for this example is illustrated by Fig. 2(A). We see that there is a well defined peak in the spectrum at about 0.14π, a "hump" at about 0.3π and fairly rapid fall off above 0.3π. Fig. 2(B) represents a smoothed periodogram (M = 128 and N = 20.48K) for the hardlimited AR series. We see from the periodogram that the peak is quite discernible but that the "hump" at 0.3π is not and that the roll off above 0.3π is not discernible. Fig. 2(C) represents the spectral estimate (M=128 and N=20.48K) obtained using the covariance transformation (6). For this case we see that the estimated peak more closely represents the actual peak, the "hump" at 0.3π is discernible and the estimate falls off above 0.3π at a rate about as rapid as the actual spectral density but with considerable variance above 0.7π. The spectral estimates of Fig. 2(B) and (C) are typical of the ten computed for this example.

Fig. 4 illustrates the results of estimating the normalized and frequency averaged error for both spectral estimates. The curves of Fig. 4 are the log of $\bar{v}^2$ for a range of sample sizes N. The top curve is the result for the smoothed periodogram, and we see no decrease in the error as N increases. The bottom curve is the result for the spectral estimate (6), and we see that the error decreases significantly as N increases.

B. The AR(2) Example - Broadband Spectral Density

The parameters chosen for this example are:

$p = 2 \quad a_1 = 0.75 \quad a_2 = -0.5 \quad \sigma_\varepsilon^2 = 1.0.$

The actual spectral density for this example is illustrated by Fig. 3(A). We see that the spectral density is very smooth with a broadband "hump" at 0.3π and a rapid roll off above 0.3π. Fig. 3(B) represents a smoothed periodogram (M = 128 and N = 20.48K) for the hardlimited AR series. We see from the periodogram that the broadband "hump" at 0.3π is discernible but that the roll off of the "tail" above 0.3π is not well represented. The estimate is biased by as much as 8 - 10 dB above the actual spectra. Fig. 3(C) represents the spectral estimate (M=128 and N=20.48K) obtained using the covariance transformation (6). For this case we see that this spectral estimate represents the actual spectral density very well for both the broadband "hump" and the "tail" roll off.

Fig. 5 illustrates the results of estimating the normalized and frequency averaged error for the spectral estimates. As in the previous example, the top curve is the result for the smoothed periodogram and there is no decrease in estimate error as N increases. The bottom curve is the result for the spectral estimate of (6) and the error again decreases significantly as N increases.

IV. CONCLUSIONS

We note that the overall magnitude of the error is smaller for the broadband example than for the narrowband example. Upon comparing the results of Figs. 4 and 5 with the asymptotic results of Fig. 1 we see that the error obtained for the narrowband example does not approach the asymptotic results, while that for the broadband example is quite close.

The simulation results indicate that for the hard limited situation reasonable finite sample size spectral estimates can be obtained for the original series. While the number of examples was limited, for both cases the spectral estimate with the sin(·) transformation applied to the covariance estimates was clearly superior to that of the standard smoothed periodogram.

REFERENCES

1. D.R. Brillinger, "Estimation of the cross-spectrum of a stationary Gaussian process from its zeros," J.R. Statist. Soc. B, 30 (1968), 145-159.

2. D.F. Gingras and E. Masry, "Spectral density estimation from nonlinearly observed data," J. Time Series Analysis (accepted for publication).

3. E.R. Rodemich, "Spectral estimates using nonlinear functions," Ann. Math. Statist., 37 (1966), 1237-1256.

4. E. Parzen, "On consistent estimates of the spectrum of a stationary time series," Ann. Math. Statist., 28 (1957), 329-348.

5. S. Cambanis, and E. Masry, "On the reconstruction of the covariance of stationary Gaussian processes observed through zero-memory nonlinearities," IEEE Trans. Inform. Theory, IT-24 (1978), 485-496.

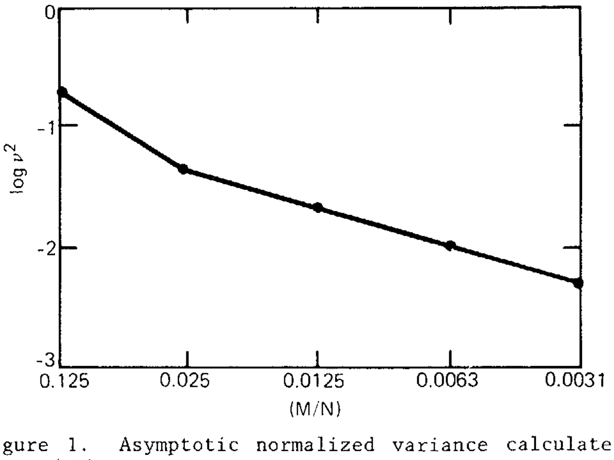

Figure 1. Asymptotic normalized variance calculated using (14).

A THEORETICAL AND EXPERIMENTAL INVESTIGATION FOR SPECTRAL ESTIMATION OF HARD LIMITED OBSERVATIONS

D.F. Gingras

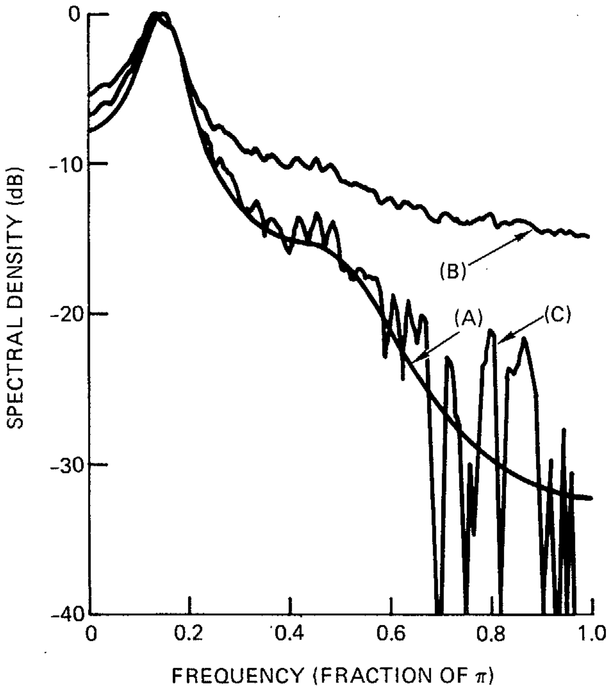

Figure 2. Spectral density and spectral estimates (M = 128 and N = 20.48 K) for the narrowband AR example: (A) exact spectral density, (B) smoothed periodogram for hardlimited series, (C) spectral estimate for hardlimited series with covariance transformation (6).

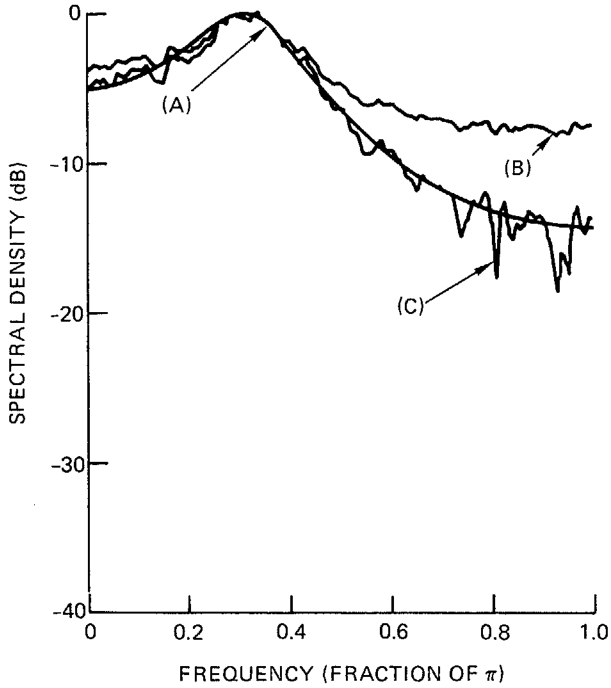

Figure 3. Spectral density and spectral estimates (M = 128 and N = 20.48 K) for the broadband AR example: (A) exact spectral density, (B) smoothed periodogram for hardlimited series, (C) spectral estimate for hardlimited series with covariance transformation (6).

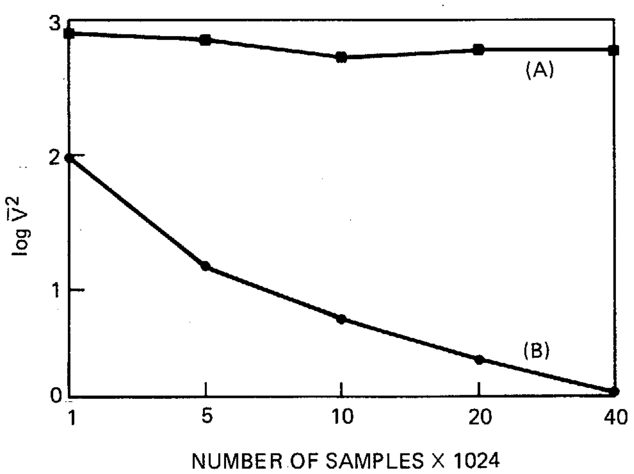

Figure 4. Normalized and frequency averaged error for the narrowband AR example (M = 128): (A) smoothed periodogram for hardlimited series, (B) spectral estimate for hardlimited series with covariance transformation (6).

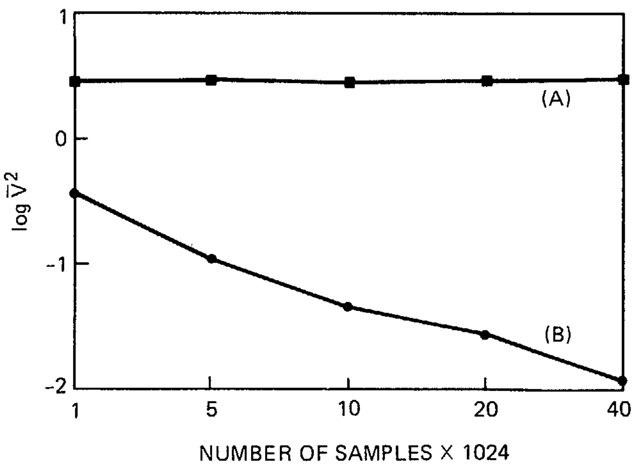

Figure 5. Normalized and frequency averaged error for the broadband AR example (M = 128): (A) spectral estimate for hardlimited series (smoothed periodogram), (B) spectral estimate for hardlimited series with covariance transformation (6).