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ON THE WIGNER TIME-FREQUENCY DISTRIBUTION

SUR LA REPRESENTATION TEMPS-FREQUENCE DE WIGNER-VILLE

Boualem BOASHASH(ex. BOUACHACHE), Lecturer
Signal Processing Group, Department of Electrical Engineering University of Queensland, 4067 Brisbane AUSTRALIA
presented by
Eric de BAZELAIRE, Expert Traitement du Signal, Department Recherches Sismiques, ELF-Aquitaine-6400 PAU, FRANCE

RESUME

Ce papier presente la contribution de l'auteur a' l'Analyse Temps-Frequence des Signaux(achevée en 1979 au Centre de Recherches d'ELF-Aquitaine, PAU). Bien que déjà' publiés au 7[e] GRETSI (1979) [7], ces resultats ont été largement incompris. Nous presentons la forme generale des Représentations Temps-Fréquence, obtenue d'apres le travail de L.Cohen in 1966 [3], montrons comment l'auteur a été amené a choisir en 1978 la Representation de WIGNER-VILLE (WVD) après une étude comparative. Les propriétés de la Representation de Wigner-Ville sont listées et un simulateur numérique(obtenu en 1978) basé sur une version numérique de la Representation de Wigner-Ville est presenté. Nous soulignons que la programmation de l'analyseur WVD est directe et aucun probleme d'Aliasing n'apparait si on respecte les bases de la theorie du signal. De plus, on montre que les artefacts créés par la methode disparaissent quand on utilise le signal analytique.

SUMMARY

This paper presents the author's contribution to Time-Frequency Signal Analysis(achieved in 1979). Although already published in French in the 7th Gretsi (1979) [Ref. 7], these results have been misused and misunderstood. We present the general form of the Time-Frequency Distributions, derived from the work of L. Cohen [3], and show how the author was led in 1978 to choose the WIGNER-VILLE Distribution, WVD after a comparative study. The properties of the WVD are listed and a digital simulator (implemented in 1978) based on a Discrete-Time Version of the WVD is presented. We emphasize that the computation of a Discrete-Time version of the WVD is straightforward and that no aliasing occurs when computed correctly Moreover, we show that low-frequency artefacts created by the method are removed by using the analytic signal.

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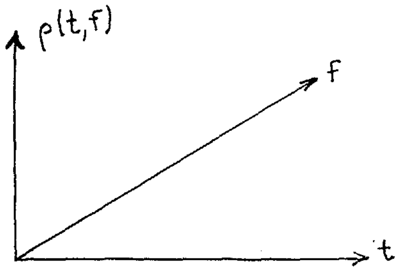
INTRODUCTION

In the case of stationary signals, i.e. signals whose spectral content does not vary with time, we can obtain an estimate of the signal energy distribution versus frequency by considering any piece of sufficient length of the signal and taking, for example, the square of its Fourier Transform modulus. The frequency resolution that is obtained depends on the length of the window used to obtain the piece of signal.

However, in many domains such as sonar, communications, geophysics,..., signals to be analysed are characterised by a variation of both amplitude and frequency with respect to time. It is obvious that, for such time-varying signals, a temporal or a spectral analysis cannot easily display suitable information.

Several different approaches have been used to deal with this situation. The Time-Frequency Distribution (TFD) in taking account of both variables, time and frequency, allows to distribute the energy evolution of the observed phenomena in a time-frequency domain and consequently gives a solution to this problem.



1. TIME-FREQUENCY DISTRIBUTIONS

1.1 Definition

Many authors have defined different time-frequency representations of signals [6]. It was then shown by L. Cohen in 1966 [3](and later by B. Escudie, and J Grea [12] [13] that all the possible definitions may be unified in a general formulation:

$$\rho(t,f)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j2\pi n(u-t)}.w(n,\tau).z(u+\frac{\tau}{2}).z^*(u-\frac{\tau}{2})\,.\,e^{-j2\pi ft}.dn.du.d\tau$$

involving a weighting function $w(n,\tau)$ such as:

$$W\varepsilon\{P\}=\{w\}(n,\tau)/w(n,\tau)=w^*(-n,-\tau);|w(n,\tau)|\leq w(o,o)=1\}$$

and where $z \in L^2(C)$ defines the analytical signal associated to the real signal $s \in L^2(R)$.

$$z(t)=s(t)+j\,H[s(t)] \quad H = \text{Hilbert Transform}$$

{P} is usually refered to as Cohen's class.

$w(n,\tau)= 1$ defines WIGNER-VILLE Distribution (WVD). A simple relationship links this latter to all other TFD's. As a general rule, the TFD has real values, positive or negative, and therefore can't be attributed any physical significance of density in the time-frequency domain. However, a condition of $\rho(t,f)$ positivity exists [6] and leads to two possible forms of $\rho(t,f)$ and their linear combinations:

- the "sonagramme" which is an envelope quadratic detection after a selective filterbank centred around the frequencies nf_0, $n=1, N$ [5].
- the "Moving Window Method" (MWM) or short Time Spectra which is a spectral analysis of a weighted slice of signal [5].

Both use the sample principle: it is an evolving analysis versus frequency in the first case and versus time in the second.

1.2 TFD and Characteristic Parameters of a Modulated Signal

$\rho(t,f)$ first order Moments yield the instantaneous frequency $f_i(t)$ and the delay time $\tau_g(f)$ of the signal if the function $w(n,\tau)$verifies the following conditions [6].

$$\left.\frac{\partial w}{\partial n}\right|_{o,o} = \left.\frac{\partial w}{\partial \tau}\right|_{o,o} = o \qquad M^1_\rho\,(f)=\tau_g(f)$$

$$w(n,o) = cte \qquad M^1_\rho(t)=f_i(t)$$

$$w(o,\tau) = cte$$

This property is very important as it enables one to obtain the Time-Frequency laws of a signal by averaging its TFD.

Unfortunately, these conditions are not fulfilled for the positive TFD's. But they are verified by the Wigner Distribution and Rihaczek Distribution [6] [9].

2. CHOICE OF A TFD

A comparative study between the most usual TFD's has been conducted [9]. The test signal used is a linear frequency modulated signal emitted by a vibrator in seismic prospecting, in the frequency range: 10Hz-250Hz.

2.1 Comparison between "Rihaczeck Analysis" and "Wigner Analysis"

Rihaczeck's Distribution of a signal s(t) is defined by: [4]

$$\rho_R(t,f)=Re\{z(t).Z^*(f).e^{-2j\pi ft}\}$$

where z(t) is the analytic signal associated with s(t), Z(f) is the Fourier Transform of z(t) and Re means real part.

The analysis of the test signal shows that the Rihaczek's TFD does not let the signal's instantaneous frequency directly appear, but offers numerous

oscillations which render the reading of the Representation very difficult [7] [9].
However, the WIGNER Distribution exhibits an amplitude concentration around the modulation law, which is so much the better than the BT product (Bandwidth Duration) of the signal is high (BT>5) [10].

2.2 Comparison between "MWM" analysis and "WIGNER Analysis

"MWM" corresponds to the most intuitive method of analyzing a signal in the time-frequency plane: it computes power spectra of weighted slices of the signal: [5].

$$\rho_{MWM}(t,f)=\left|\int_{-\infty}^{+\infty} z(u).P(u-t).e^{-j2\pi fu}.du\right|^2$$

By definition, such a short-time analysis is local and its resolution is fixed by the analysis width. Moreover, it is well-known that the analysis induces a bias on the estimation of the time-frequency laws of the signal [9] [10].
Isoamplitude curves $\rho_W(t,f)$=cte are ellipses, the major axis of which characterises the time-frequency laws of the signal for values of the BT product higher than 5 [10]. Isoamplitude curves $\rho_{MWM}(t,f)$=cte are also ellipses, but their major axis direction and resolution depend on the equivalent analysis duration Δ. The optimum analysis width is defined by the relaxation time $\Delta dfi/dt$. However, even in this optimum case, MWM resolution remains less accurate than WD's [10].

2.3 The Choice

Thus, the Wigner Distribution appears as a fruitful representation of signals in the time-frequency plane, as it has a better resolution and is independent of any analysis width.
We have therefore built an interactive simulator of time-frequency analysis using the WIGNER Distribution able to analyze any signal [6] [9].

3. PROPERTIES OF THE WIGNER-VILLE DISTRIBUTION

It is defined by:

$$W(t,f)=\int_{-\infty}^{+\infty} z(t+\tfrac{\tau}{2}).z^*(t-\tfrac{\tau}{2}).e^{-j2\pi f\tau}.d\tau$$

Previous studies have shown that the WIGNER DISTRIBUTION exhibits some very interesting properties with regard to time-frequency signal analysis [7] [8].
Here are the most important:

- $W(t,f)$ is real for all (t,f)
- spectral density: $\int_{-\infty}^{+\infty} W(t,f).dt=|Z(f)|^2$
- instantaneous power: $\int_{-\infty}^{+\infty} W(t,f).df=|z(t)|^2$
- signal energy: $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(t,f).dt.df=E_z$
- if $z(t)=o$ for $t<T_1$ and $t>T_2$, then $W(t,f)=o$ for $t>T_1$ and $t>T_2$
- if $Z(f)=o$ for $f<f_1$ and $f>f_2$, then $W(t,f)=o$ for $f<f_1$ and $f>f_2$
- the first-order Moments of the WD given fi(t), instantaneous frequency and $\tau_g(f)$, group delay
- if $y(t)=s(t)*h(t)$, then we have: $W_y(t,f)=W_s(t,f)\overset{*}{(t)}W_h(t,f)$
  *= convolution
- reversibility of the Wigner Distribution: we can reconstruct s(t) as follows: [8]

$$\int_{-\infty}^{+\infty} W(\tfrac{t}{2},f).e^{j2\pi ft}.dt=z(t).Z^*(o).$$

Thus, the WD contains and conveys all the information vehicled by the signal. No information is lost.
For the discussion of those properties, see [8] and [14]. In the most general case, an accurate study of WD needs simulations. That is why numerical studies have been done using the Fast Fourier Transform [7] [8].
A theoretical study has explained in each case the WD behaviour.

4. COMPUTATION OF THE WIGNER DISTRIBUTION

4.1 Introduction

The computation of its discrete-time version was achieved by the author in 1978 [6], published first in 1979 [7] and explained again in several international conferences [10].
We emphasize here that the Wigner Distribution is straightforward to compute, and that no aliasing is created by the method.

4.2 Why Aliasing is not a problem

The Wigner Distribution is expressed as follows:

$$W(t,f)=\int_{-\infty}^{+\infty} z(t+\tfrac{\tau}{2}).z^*(t-\tfrac{\tau}{2}).e^{-j2\pi f\tau}.d\tau$$

$$=F.F.\{z(t+\tfrac{\tau}{2}).z^*(t-\tfrac{\tau}{2})\}$$

(F.T. = Fourier Transform)

where z(t) is the analytic signal associated with the real signal s(t) as follows:

$$z(t)=s(t)+j.H[s(t)],$$

H=Hilbert Transform

s(t) is assumed to be time-limited and to have a large BT (Bandwidth x Duration) Product.
In practise, the analytic signal is calculated in the frequency domain, using the following property

$$Z(f)=2.\ S(f) \quad , f \text{ positive}$$
$$= o \text{ otherwise}$$

where Z(f) and S(f) are respectively the Fourier Transforms of z(t) and s(t). Its discrete time version is: $W(n.\Delta t,k.\Delta f)=W(t,f)$ where t and f are respectively the time sampling interval and the frequency sampling interval chosen correctly.

$$W(n,k)=\sum_{m=-\infty}^{+\infty} z(n+\frac{m}{2}).z^*(n-\frac{m}{2}).e^{-j2\pi km}$$

$$= D.F.T.[z(n+\frac{m}{2}).z^*(n-\frac{m}{2})].$$

The current variable here is now $\frac{m}{2}$ and n can be considered as a shift (or delay). The D.F.T. goes from time m to frequency k.

Using the scale property of the F.T. Yields

$$W(n,k)=D.F.T.[z(n+\theta).z^*(n-\theta)]$$

$$\theta \ - >2k$$

Suppose now that f_{max} is the highest frequency component of the incoming signal. Obviously(it is only a scaling property of the Fourier Transform, discrete on continuous),the calculation of the WD requires the knowledge of frequencies up to $2.f_{max}$ for $z(\frac{\tau}{2})$, but is quite compatible with the knowledge of frequencies up to f_{max} for $z(\tau)$.

*another interpretation

Suppose we have the sequence x(n) sampled at twice the nyquist rate, N samples. All information is contained in these samples. However, a characteristic of the Wigner distribution is that it requires the knowledge of the values inbetween $x(n+\frac{1}{2})$. To solve this problem an obvious solution is oversampling. But it is not necessary because another solution is interpolating. Our program incorporates an automatic interpolation in the calculation of the analytic signal without loosing or adding any information, by adding N zeroes to the result of the F.F.T. of the original real signal. The computer program works then on an analytic signal with 2N samples.

4.3 Importance of the Analytic Signal

1)Finite Duration: In general, if s(t) is time-limited, there is no reason why z(t) should have the same property. However, the signals considered are assumed to have a large B.T. product (B=bandwidth, T=duration). In this case, if the real signal is:

$$s(t)=a(t).\cos\Phi(t)$$

the use of Bedrosian's theorem yields the following approximation: [5], [15]:

$$z(t)=s(t)+j.H[s(t)]=a(t).e^{j\Phi(t)}$$

where H denotes the Hilbert Transform.

Then if s(t) is time-limited, and if B.T. is large, then a time-limited z(t) is a reasonable approximation. Moreover, it has to be said that in any case, the real part of the analytic signal z(t) remains(obviously) time-limited.

2)Artefacts in the Wigner Distribution: When one does not use the analytic signal z(t), but the real signal s(t).

We showed as early as 1979 that the Wigner Distribution creates low-frequency artefacts in the (t,f) domain when one does not use the analytic signal [9]*. They are caused by cross-products between positive and negative frequencies. This is illustrated in Figure 3 and 4 of the paper Eusipco 1983 [16]

5. WIGNER ANALYSIS OF MODULATED SIGNALS

The analysed signals are real, causal, almost time-and-bandlimited, of finite energy, centered (S(f=o)=o) and verify BEDROSIAN's conditions (BT>>1). They are expressed by: $s(t)=a(t).\cos\Phi(t)$, in which case, the analytical signal associated z(t) is expressed as:

$$z(t)=a(t).e^{j\phi(t)}$$

and therefore the instantaneous frequency defined by:

$$f_i(t)=\frac{1}{2\pi}\cdot\frac{d\phi}{dt}(t)$$

has a physical sense: it defines the frequency modulation law of the signal.

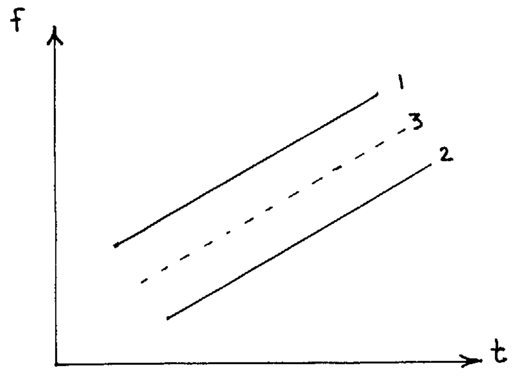
5.1 WD of Monocomponent signals

We call monocomponent signal any signal for which the instantaneous frequency law is invertable, so that the law $t=f_i^{-1}(f)$ is not multiform. This latter represents then the group delay of the signal. In this case, the WD makes the frequency modulation law of the signal appear in the time-frequency domain, by visual correlation of maximum amplitude curve and can be estimated simply by peak detection.

5.2 WD of Multicomponent signals

Such a signal is characterized as the sum of several monocomponent signals. The WD's behaviour depends on whether the frequency modulation laws of each component have same gradient or not.

- if these gradients are different(non-parallelism in the (t,f) plane), WD analysis allows to separate the characteristics of each component in the time-frequency domain.
- If these gradients are equal(parallelism in the (t,f) plane), WD analysis creates artefacts (a ghost law 3 appears equidistant between the two real laws 1 and 2). However, this structure can be used for the inerpretation of time-frequency analysis.



5.3 Noise Influence on the WVD

We have studied the behaviour of the WVD of a signal in additive noise, with different ratios S/N. s(t) is a signal with a linear frequency modulation law and n(t) is a random guassian white noise is the same spectral bandwidth. The study shows that the WVD has a good behaviour in the presence of noise. It shows the importance of such an analysis, as classical methods are unable to estimate frequency modulation laws in such cases [9].

6. CONCLUSION

This paper shows that the WIGNER DISTRIBUTION is the best tool for a time-frequency analysis:
- it allows a good visualisation of the signal time-frequency law.
- it does not depend on any window-width defined a priori.
- its execution is relatively easy.
- its computation is straightforward.

The property of reversibility between s(t) and its WD allows filtering and pattern recognition in the time-frequency domain [11].
This method has been recently applied by the author to detect nitrogen bubbles moving in the blood(produced in a hyperbaric environment) [11]. Previously, the method has been developed by the author within ELF-AQUITAINE Geophysical Research Centre, Pau, France and has been applied to estimate the absorption and dispersion effects of the earth from a Vertical Seismic Profile [9]. This study has led up to perform an automatic estimation of the absorption parameters and has turned out to be very useful for the stratigraphic interpretation.

*The work published by P. FLANDRIN et al., in GRETSI, SIGNAL PROCESSING and Comptes Rendus Acad.Sciences, between 1979 and 1984 is mainly the rewriting of the author's work(published in [6],[7],[9])after the author left ICPI-LYON .

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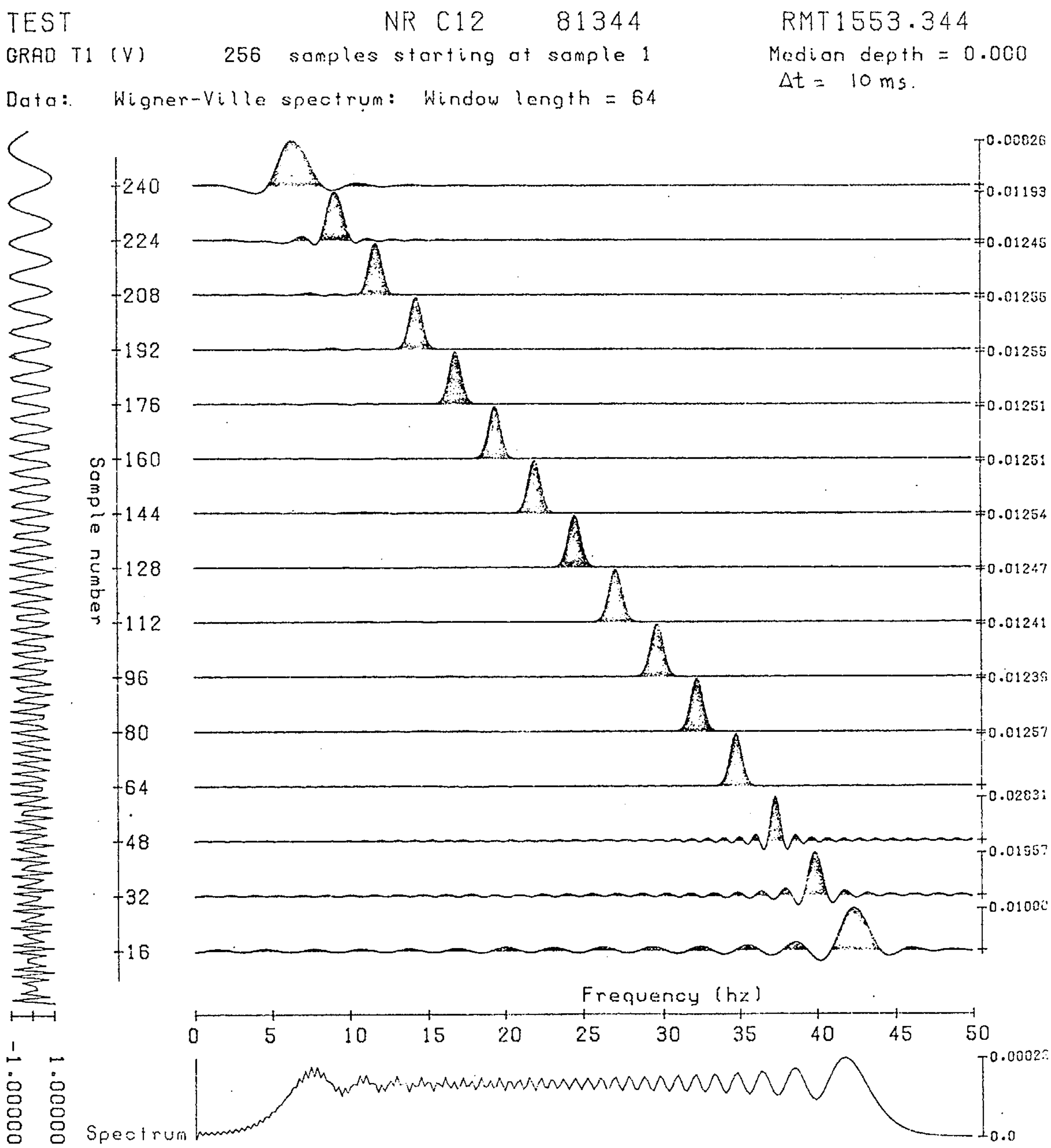
Note: Boualem Boashash was previously spelled Bouachache. This work was funded between 1978 and 1983 by ELF-Aquitaine,Geophysical Research Center, PAU, FRANCE.

APPENDIX: THE WVD OF A TEST SIGNAL



This figure shows the WVD of a test signal (Frequency modulated signal with a linear FM law), calculated by the discrete-time analyser implemented by the author in 1978 for ELF-Aquitaine Geophysical Research Center, PAU, FRANCE (References [6],[7],[9]).