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Digital filters for differentiating signals of low accuracy.

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RESUME

Dans ce travail quelque filtres pour differentiation des signaux en présence de bruit sont analyses.
Les fonctions de transfert sont calculés pour les filtres proposées.

SUMMARY

The paper analyzes some high frequency suppression, differentiating digital filters. The filters are equal spaced.
Transfer functions are calculated and shown for the proposed filters.

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Introduction

For many signal processing applications is it usefull to differentiate a signal. Numerical differentiation is often hazardous when the signal is contaminated with noise. The reason is that the amplitude of the transfer function for differentiation is proportional to the frequency, which means that high frequencies in the signal and noise spectrum will be amplified more than the low frequencies. This may ruin a differentiation procedure if high frequencies in the noise spectrum are more predominant than high frequencies in the signal spectrum. The obvious thing to do is to use a differentiation procedure which cuts off the undesired high frequency part of the spectrum. Unfortunately,litterature about such procedures is almost nonexistant.

The present paper presents a comparison between some high frequency suppression formulas of the form:

$$f'_o=(a(f_1-f_{-1})+b(f_2-f_{-2})+c(f_3-f_{-3}))/h \qquad (1)$$

where h is the sample spacing.

Three types of filter formulas has been investigated:

1) Lagrange differentiation formula through 3 points (b=c=o), 5 points (c=o), and 7 points. With Lagrange differentiation formula one calculates the polynomial of degree 2N passing through 2N+1 points and differentiate this polynomial to find the coefficients a, b, and c in eq.(1).

2) Differentiation of orthogonal (or Gram) polynomials passing between 5 and 7 points. For this filter one calculates the polynomial of degree M which in a least-squares sense passes between 2N+1 points ($M \leq 2N$). This polynomial is also differentiated to give the coefficients. N.B. When M=2N the result is the same as one obtains with Lagrange differentiation formula.

3) Differentiation of Gaussian smoothing formulas (developed by the author) through 5 and 7 points. The data values are smoothed by convolution of a Gaussian distribution and thereafter differentiated.

Determination of coefficients

1) The coefficients in Lagranges differentiation formula can be determined from differentiation of Lagranges interpollation formulas (see e.g.Abramowitz p.882). It is mentioned above that the coefficients can be determined as a special case of the orthogonal polynomials where M=2N (please look below).

2) If $p_j(s)$ is a polynomial of degree j, a least squares approximation of degree M for the data values f_s at the points $x_s = x_o+sh$ where s = -N,...,-1,o,1,...,N can be written as:

$$y_M(s)=\sum_{j=0}^{M} b_j\, p_j(s) \qquad (2)$$

with

$$b_j=\frac{\omega_j}{\gamma_j} \qquad (3)$$

where

$$\omega_j=\sum_{s=-N}^{N} f_s\, p_j(s) \qquad (4)$$

and

$$\gamma_j=\sum_{s=-N}^{N} (p_j(s))^2 \qquad (5)$$

Differentiating (2) and inserting (3), (4), and (5) gives for the point s=o the approximation:

$$y'_M(0)=\sum_{s=1}^{N}(f_s-f_{-s})\left(\sum_{j=0}^{M} p_j(s)\,\frac{p'_j(0)}{\sum_{t=-N}^{N}(p_j(t))^2}\right)\cdot\frac{1}{h} \qquad (6)$$

It is possible to show (Ralston p.259) that the polynomials (called Gram polynomials) can be found from the recurrence formula:

$$p_{j+1}(s)=s\,\frac{\varepsilon_{j+1}}{\varepsilon_j}\,p_j(s)-\beta_j\,\frac{\varepsilon_{j+1}}{\varepsilon_{j-1}}\,p_{j-1}(s) \qquad (7)$$

where

$$\beta_j = \frac{j^2\left((2N+1)^2 - j^2\right)}{4(4j^2-1)} \tag{8}$$

and

$$\varepsilon_j = \frac{(2j)!\,(2N-j)!}{(j!)^2\,(2N)!} \tag{9}$$

Inserting eqs:(7), (8), and (9) into eq.(6) gives the desired coefficients.

3) Differentiation by Gaussian smoothing is done as proposed by the author (Sjøntoft, p.2o1), where it is shown that if the Dirac-delta function is approximated by a Gaussian function of spread s one gets the approximation for the derivative:

$$f'(x) = \lim_{s \to 0}\left[\int_{-\infty}^{\infty} \frac{t}{s^3\sqrt{2\pi}} \exp\left(\frac{-t^2}{2s^2}\right) f(x+t)\,dt\right] \tag{1o}$$

If one calculates the value of the expression:

$$c_i = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{t}{s^3\sqrt{2\pi}} \exp\left(\frac{-t^2}{2s^2}\right) dt \tag{11}$$

one gets

$$f'(0) = \frac{1}{h}\sum_{i=1}^{N} c_i\left(f_i - f_{-i}\right) \tag{12}$$

The degree of smoothing is determined by the value of s used in eq. (11).

Values of the coefficients calculated by the three methods is shown in the table:

	a	b	c
Lagrange 7 points	o.75o	÷.15o	o.o17
-"- 5 points	o.667	÷.o83	---
-"- 3 points	o.5oo	---	---
Gram 1.deg. 7 points	o.o36	o.o71	o.1o7
-"- 3.deg. 7 points	o.23o	o.266	÷.o87
-"- 1.deg. 5 points	o.1oo	o.2oo	---
Gauss s=1.5 7 points	o.1o6	o.111	o.o57
-"- s=1.2 7 points	o.159	o.119	o.o35
-"- s=1.o 7 points	o.244	o.113	o.o17
-"- s=1.o 5 points	o.249	o.125	---
Gauss s=o.8 5 points	o.332	o.o84	---
-"- s=o.6 5 points	o.441	o.o29	---

Transfer functions

To find the transfer function for a filter given by eq.(1) one inserts a pure sinusoid $f(t) = e^{i\omega t}$ to see how it is treated by the filter. We see that a differentiation should give $f'(t) = i\omega e^{i\omega t}$.
If we put t = kh we have $f_k = e^{i\omega kh}$
Inserting into eq.(1) gives:

$$f'_k = i\omega e^{i\omega kh}\left(2a\frac{\sin \omega h}{\omega h} + 4b\frac{\sin 2\omega h}{2\omega h} + 6c\frac{\sin 3\omega h}{3\omega h}\right) \tag{13}$$

If we define the transfer function as

$$g(\omega) = \frac{f'_k}{f_k} \tag{14}$$

when $f_k = e^{i\omega kh}$, we have for differentiation $g(\omega) = i\omega$ and for the filter in eq.(1)

$$g(\omega) = i\omega\cdot 2\cdot\left(a\frac{\sin\omega h}{\omega h} + 2b\frac{\sin 2\omega h}{2\omega h} + 3c\frac{\sin 3\omega h}{3\omega h}\right) \tag{15}$$

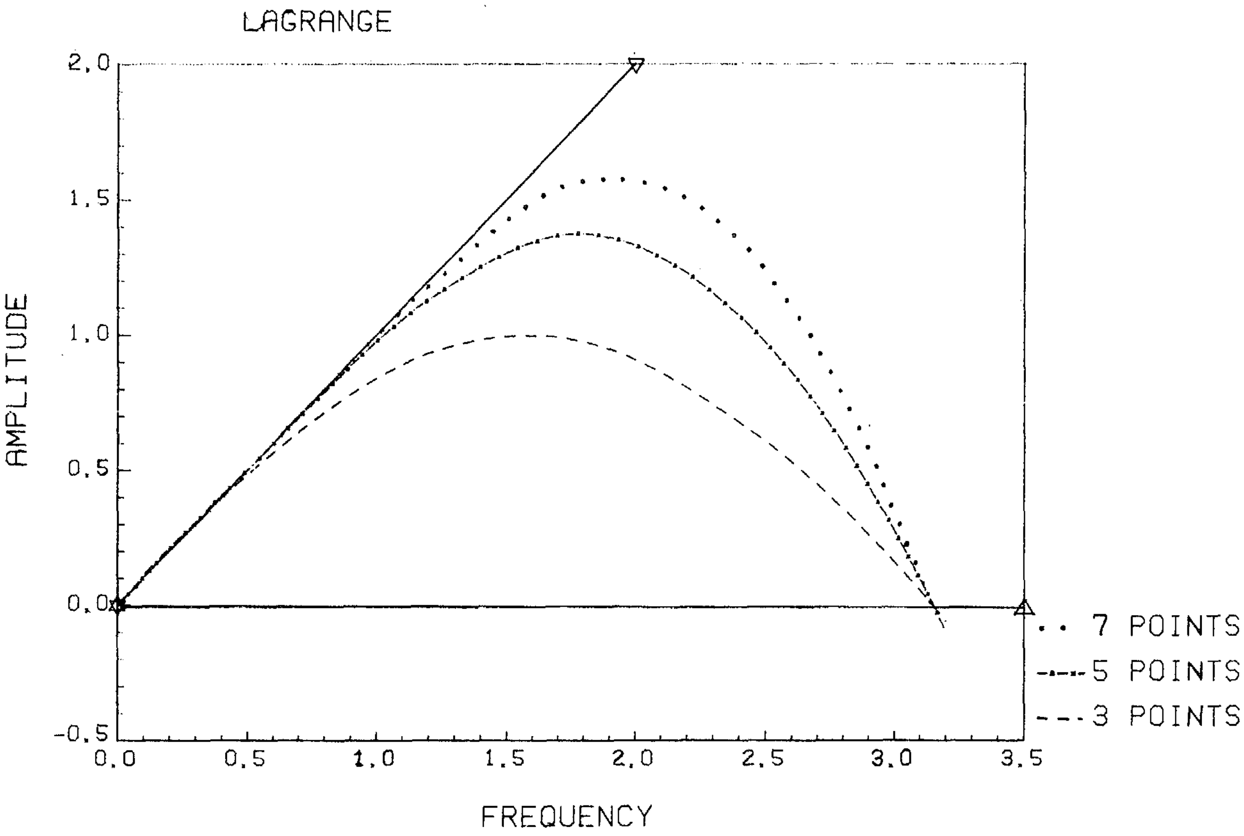
We note that

$$\lim_{\omega \to 0} g(\omega) = i\omega\cdot 2\cdot(a+2b+3c) \tag{16}$$

If a filter is to be called differentiating for $\omega \to 0$ we must have:

$$a+2b+3c = 0.5 \tag{17}$$

We see that the filter coefficients in the table fulfils this condition. The figures show the imaginary amplitudes of the transfer functions for the filters proposed in this paper.


LAGRANGE
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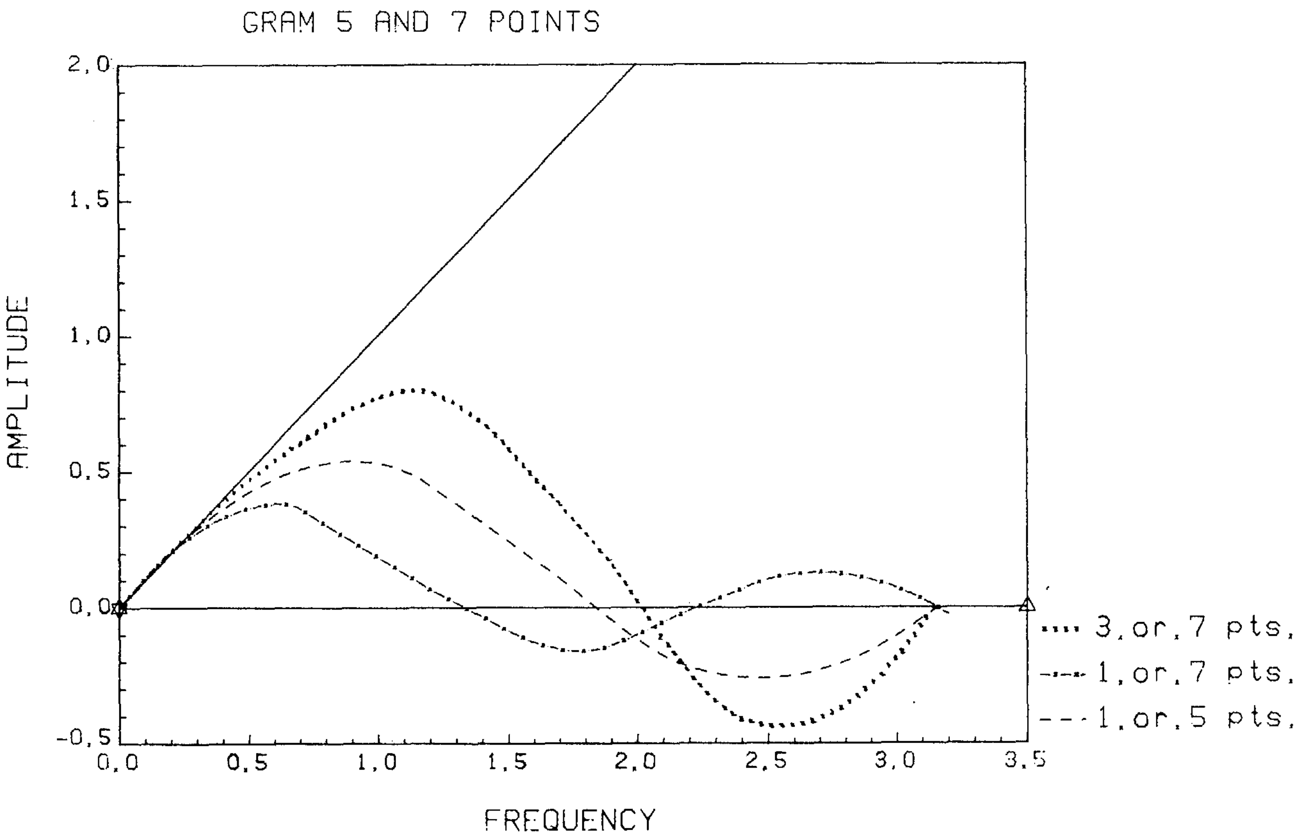
1,5
1,0
0,5
0,0
-0,5
AMPLITUDE
0,0
0,5
1,0
1,5
2,0
2,5
3,0
3,5
FREQUENCY
7 POINTS
5 POINTS
3 POINTS


GRAM 5 AND 7 POINTS
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1,5
1,0
0,5
0,0
-0,5
AMPLITUDE
0,0
0,5
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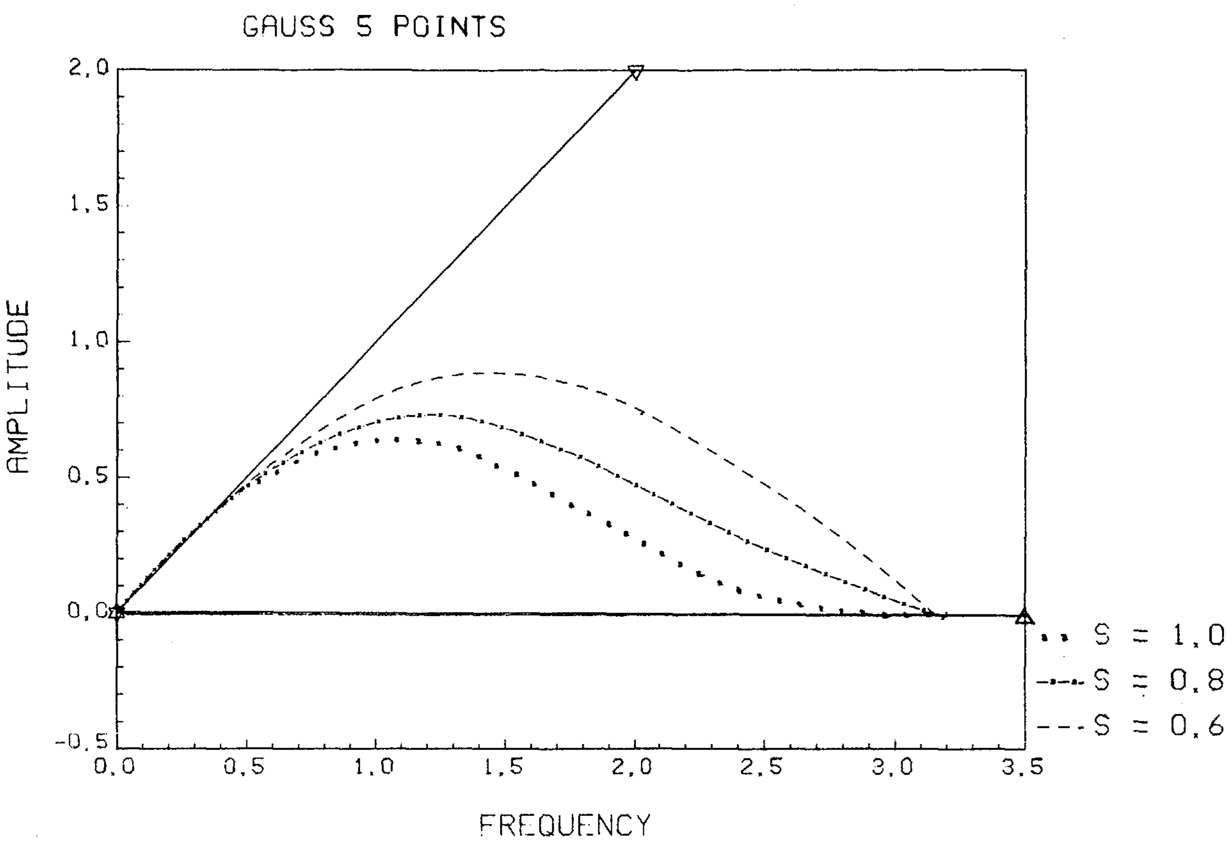
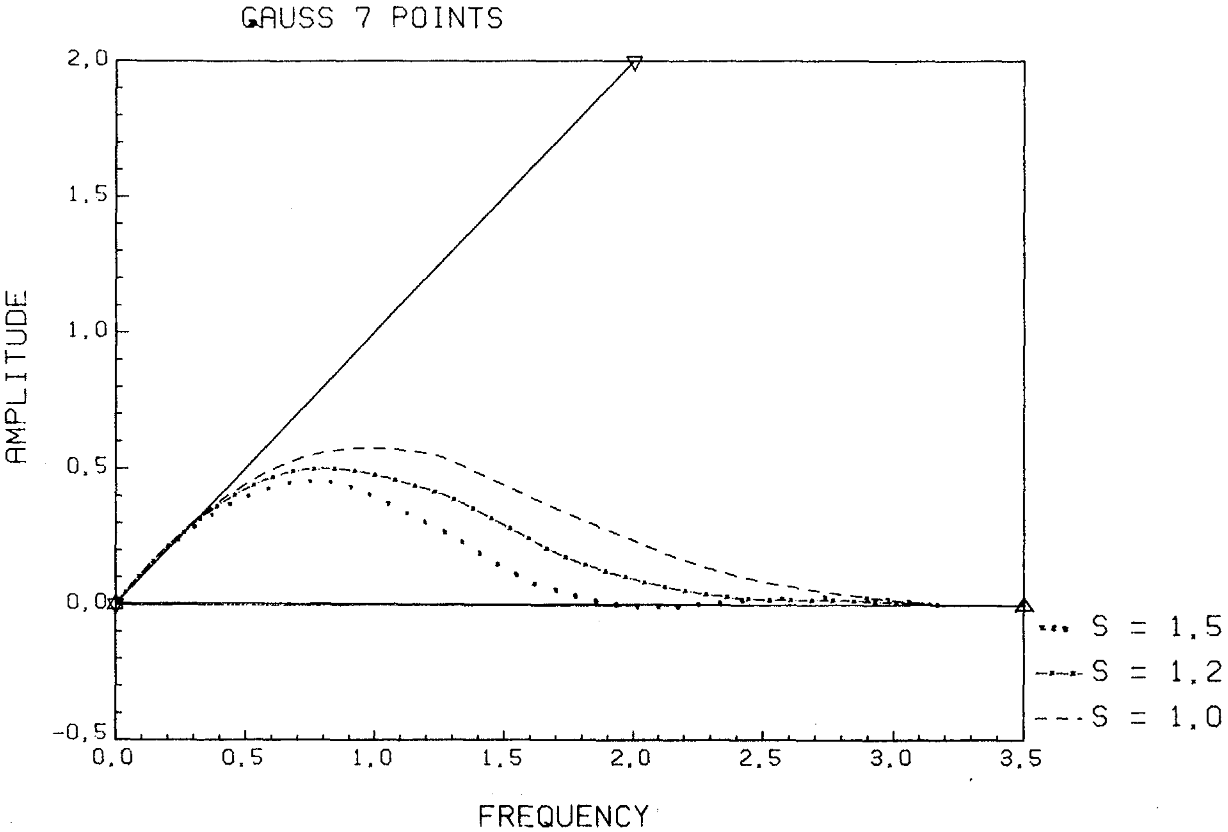
1,5
2,0
2,5
3,0
3,5
FREQUENCY
3,or,7 pts,
1,or,7 pts,
1,or,5 pts,

Digital filters for differentiating signals of low accuracy.





Discussion

It is evident from the transfer functions shown in the figures, that it is possible to design a differentiating filter which suppresses high frequencies better than the usually proposed Lagrange filter formulas. The Gaussian smoothing formulas seems to give the freest choice in the degree of smoothing. It is also possible to avoid the ringing effect which exist in the Gram polynomial filters.
The methods described above to analyze and construct filters can easily be extented to more than 7 points.

References

Abramowitz and Stengun, Handbook of Math. Functions, Dover 1975.

Ralston and rabinowitz, A first course in num. anal., Internat.Stud. Ed.,1978.

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