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DEVELOPMENT OF A NEW SIGNAL PROCESSING TOOL USING A COMPLEX EXPONENTIAL TECHNIQUE

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RESUME

Une technique par exponentielle complexe a été développée pour représenter et analyser des signaux limités dans le temps, et l'évaluation de cette méthode par comparaison avec les techniques plus conventionnelles par transformation discrète de Fourier, est discutée. On montre que pour une durée donnée d'un signal contenant une information discrète en fréquence, échantillonnée au moins au rythme de Nyquist, l'algorithme par exponentielle complexe peut souvent fournir une meilleure résolution en fréquence que les techniques standard de Fourier. On montre aussi que l'algorithme par exponentielle complexe apparaît être un meilleur mécanisme pour l'interpolation entre points d'un signal discret contenant des composantes discrètes de fréquence, en présence de bruit large bande. Les effets du bruit dans la technique par exponentielle complexe et les difficultés de calcul liées au présent algorithme sont aussi analysés. Enfin, on discute les applications possibles de cette technique par exponentielle complexe à d'autres domaines comme la résolution angulaire fine de cibles multiples, l'estimation de Doppler etc...

Cet article rassemble les résultats des recherches de ces dernières années et il est de nature didactique.

SUMMARY

A complex exponential technique developed for the representation and analysis of time-limited signals is defined, and its evaluation with respect to conventional discrete Fourier techniques is discussed. It is shown that for a given length of a signal containing discrete frequency information, sampled at least to the Nyquist criterion, the complex exponential algorithm can often provide increased frequency resolution over standard Fourier techniques. It is also shown that the complex exponential algorithm provides an improved mechanism over Fourier techniques for interpolation between points in a sampled signal containing discrete frequency components in the presence of broadband noise. The effects of noise in the complex exponential technique and the computational difficulties associated with the present complex exponential algorithm are also discussed. Applications of this complex exponential technique in other areas such as high angular resolution of multiple targets, Doppler estimation, etc., are discussed. This paper summarizes research results of the last few years' and is of a tutorial nature.

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INTRODUCTION

Fourier techniques are the most widely used and valuable general-purpose signal-analysis techniques available today for digital signal processing. The advent of what has become known as the fast Fourier transform has made real-time digital signal analysis a practical reality. However, several stringent assumptions concerning the signal to be analyzed and the associated analysis parameters are necessary in order to effectively use digital Fourier techniques for signal analysis. Often these assumptions result in a degradation in certain performance characteristics of Fourier techniques, for example, the ability to resolve discrete frequency lines in time-limited signals.

Fourier techniques, in particular, involve the assumption that a signal is either infinite in duration or is repetitive over all time within some fundamental period. However, in a practical digital analysis it is necessary to truncate the signal. Consequently, the analysis of a pure sine wave using Fourier techniques typically results in a (sinx)/x, as shown in figure 1, that limits the ability of Fourier techniques to resolve two closely spaced discrete frequency components in a signal. Normally two discrete frequency components cannot be resolved using Fourier techniques unless a segment of data of length T is utilized, where T is the reciprocal of the frequency difference between the two discrete components, as shown in table 1. As noted, this expression applies even at high signal-to-noise ratios (SNR's).

Table 1. Resolving two closely spaced discrete frequency components using Fourier techniques

Signal $\equiv f(t) = A_1 \sin(2\pi f_1 t + \phi_1) + A_2 \sin(2\pi f_2 t + \phi_2)$

To resolve these two discrete frequency components using Fourier techniques requires a segment of data length T, where

$$T \geq \left| \frac{1}{f_2 - f_1} \right| .$$

This expression applies even at high SNR's.

It is also necessary when using Fourier techniques to make an a priori specification of the bandwidth structure of the analysis; that is, the signal is analyzed in bandwidths that are predetermined and normally equally spaced. The signal energy is simply projected on a set of functions of the form $\cos n w_0 t$, where w_0, the analysis bandwidth, is predetermined. The bandwidth structure is determined by the parameters of the analysis, such as sampling rate and number of data points, and has little relationship to the true frequency content of the signal. Consequently, many of the frequency values computed are typically of little interest.

One technique that has been particularly successful for signal analysis is the complex exponential technique. In the early 1960's, this technique was utilized by Huggins[1] and McDonough,[2] of The Johns Hopkins University, and others[3] to reduce the number of parameters required to represent an oscillatory signal to some desired accuracy. We have found the technique to be particularly useful for the analysis of time-limited signals containing discrete frequency components. The complex exponential signal representation for a signal length T is defined in figure 2.

Here we review the complex exponential technique with examples and explore possible application in many areas such as estimation of Doppler shift, bearing estimation of multiple targets without forming beams, etc., and cite future research areas.

THEORETICAL APPROACH

The complex exponential representation is defined as

$$f(t_k) = \sum_{j=1}^{2M} A_j \, e^{s_j t_k}, \quad k = 0, 1, \ldots \ldots 4M - 1; \qquad (1)$$

that is, for a complex exponential representation, the sequence of sampled-data points $\{f(t_k)\}$ is represented as a summation over 2M basis functions of the complex exponential type. The sequence for coefficients $\{A_j\}$ and $\{s_j\}$ are to be determined using the known sampled-data sequence $\{f(t_k)\}$. It can be shown that 2M is equal to one-half the number of input digital data points, and an exact solution to the system of nonlinear equation 1 for the sequence $\{A_j\}$ and $\{s_j\}$ is possible. If 2M is less than one-half the number of input points, a least-squares solution of the nonlinear system of equation 1 can be found.

The Prony technique is the standard computational method for obtaining a complex exponential representation.[3] This algorithm consists of, first, solving a linear Toeplitz system, as shown in table 2. Next, a polynomial equation is solved for its complex roots $\{x_j\}$, where

$$x_j = e^{s_j \Delta t} ,$$

and Δt is the sampling period associated with the digital data as shown in table 3.

Table 2. Solving a linear Toeplitz system of equations for the unknown parameters a_j

$$\begin{bmatrix} f_{2M-1} & f_{2M} & f_{2M+1} & \cdots & f_{4M-2} \\ f_{2M-2} & f_{2M-1} & f_{2M} & \cdots & f_{4M-3} \\ f_{2M-3} & f_{2M-2} & f_{2M-1} & \cdots & f_{4M-4} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ f_0 & f_1 & f_2 & \cdots & f_{2M-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{2M-1} \end{bmatrix} = \begin{bmatrix} f_{4M-1} \\ f_{4M-2} \\ f_{4M-3} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{2M} \end{bmatrix}$$

Table 3. Solving polynomials and calculating the complex root to get the frequency component

Solve a polynomial equation for its complex roots x_j:

$$\sum_{j=1}^{2M} a_j (x)^j = 0 .$$

Calculate the complex s_j from the relation

$$x_j = e^{s_j \Delta t} , \quad j = 1, 2, \ldots , 2M .$$

This yields the frequencies and real exponential modulation factors, since

$$s_j = x_j + i\omega_j .$$

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Once the complex roots that provide the frequency component are determined, the complex amplitudes $\{A_j\}$ are calculated from the solution of a linear Van der Monde system of the form shown in table 4.

Table 4. Solving a linear Van Der Monde system of equations for the complex amplitudes A_j

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_2 & X_2 & \cdots & X_{2M} \\ X_1^2 & X_2^2 & X_3^2 & \cdots & X_{2M}^2 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ X_1^{2M-1} & X_2^{2M-1} & X_3^{2M-1} & \cdots & X_{2M}^{2M-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_{2M} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{2M-1} \end{bmatrix}$$

EXAMPLES

Consider a signal consisting of two sine waves of amplitude 1, initial phase zero, and frequencies of 560 and 700 Hz. A sampling rate of 0.1 millisecond is used. Since these two signal components differ in frequency by 140 Hz, Fourier techniques require a minimum of 7.1 milliseconds of signal to resolve the two signal components. The results of both discrete Fourier and complex exponential spectral analysis of the defined signal using signal durations of 3.1 and 6.3 milliseconds are shown in figure 3.

In the no-noise case, above, the complex exponential results are accurate to about five significant figures. Consequently, it can be seen that the complex exponential performance is substantially better than the Fourier results for the example shown. In particular, in the lower half of figure 3, note how the superposition of sidelobes in the Fourier transform tends to make the frequency results of the Fourier transform very inaccurate.

APPLICATIONS OF THE COMPLEX EXPONENTIAL TECHNIQUE

The complex exponential technique has possible applications in the following areas:

- Analysis of transient signals
- Analysis of time-limited signals
- Estimation of Doppler shift of a continuous wave (CW) pulse
- Simultaneous bearing estimation of multiple targets without forming beams
- Possible application in bearing and range estimation.

Examples of time-limited signals such as transient signals have been discussed. Here we shall discuss the problems of bearing estimations of different targets using a linear array as shown in figure 4. The output of each hydrophone consists of signals arriving from several plane waves. The output of the mth hydrophone may be described as

$$E_m = \sum_{j=1}^{L} W_j \, e^{\frac{i2\pi}{\lambda} dm \sin\theta_j} \; ; \; m = 0,1,2, \ldots , M$$

$$= \sum_{j=1}^{L} W_j x_j^m \, ,$$

where $x_j = e^{\frac{i2\pi}{\lambda} d \sin\theta_j}$, d is the element spacing, L is the total number of incoming signals, θ_j is the angle of arrived signal j. One wishes to determine θ_j. The above equation can be solved the same way as equation 1. If M is equal or greater than 2L, the above equation may be solved to obtain $\{\theta_j\}$.

FUTURE RESEARCH

Preliminary simulation results indicate that at high SNR's (> 20 dB) the complex exponential technique works well. However, it needs further investigation to determine the influence of the following on the complex exponential technique:

- Effects of noise
- Effects of initial phase of each tone
- Minimum required length of signal
- Unequal data samples
- Improvement of SNR by averaging.

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3. F. R. Spitznogle and A. H. Quazi, "Representation and Analysis of Time-Limited Signals Using a Complex Exponential Algorithm," Journal of the Acoustical Society of America, Vol. 47, No. 5 (Part 1), May 1970, pp. 1150-1155.

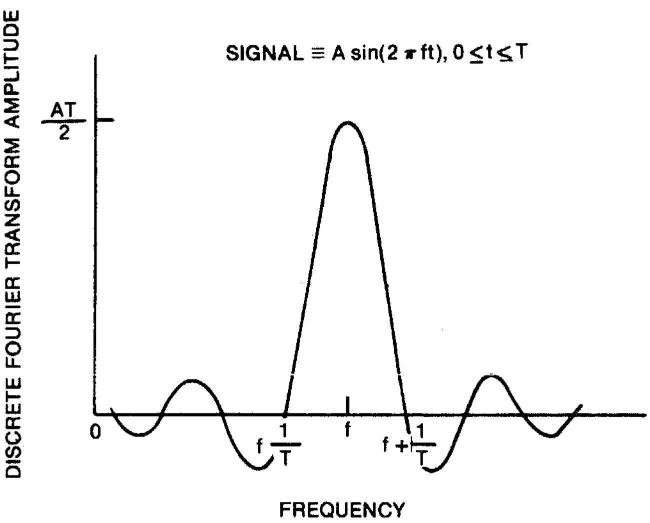

Fig. 1. Analysis of pure sine wave using Fourier techniques

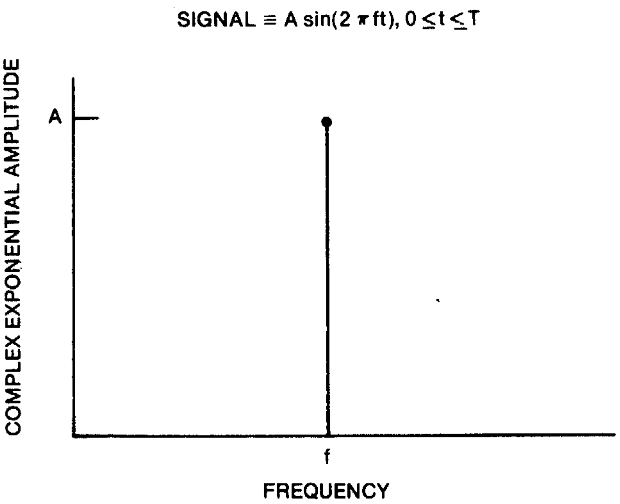

Fig. 2. Complex exponential technique

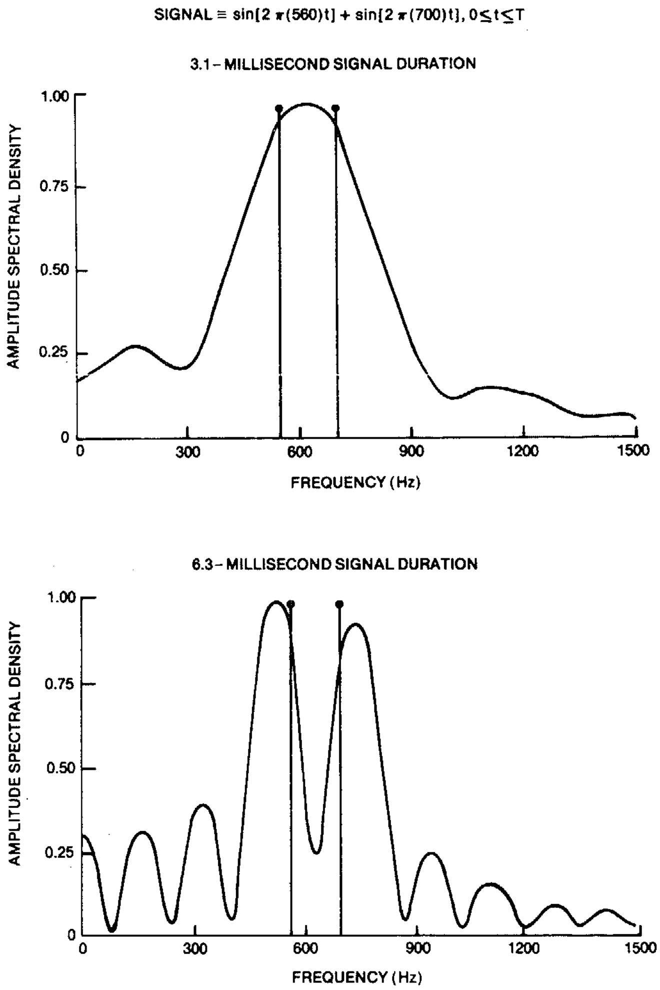

Fig. 3. Comparison of Fourier and complex exponential techniques for spectral analysis

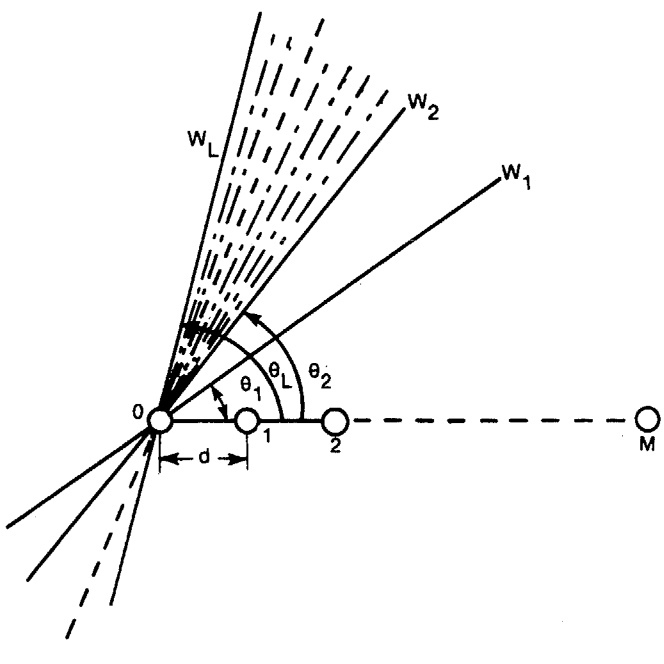

Fig. 4. Bearing estimations of multiple targets