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STATISTICS OF TRANSFORMED COHERENCE ESTIMATES

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RESUME

Les résultats précédemment acquis sur les statistiques de l'estimation du carré de la grandeur cohérente $\hat{C}$ et de l'estimation de la grandeur cohérente $\sqrt{\hat{C}}$ ont exigé la simplification des fonctions hypergéométriques ${}_3F_2$, parfois au prix de beaucoup de mal. On peut désormais éviter ces conditions, en recourant à des relations relativement simples qui donnent, par exemple, la déviation, la variation et l'erreur de la moyenne quadratique de toute transformation non linéaire de $\hat{C}$. Ces relations impliquent uniquement le nombre N (le nombre d'éléments de données à partir desquelles on établit la moyenne de l'estimation de la grandeur cohérente) et la transformation non linéaire et ses deux premières dérivées, évaluées à la valeur réelle C de la grandeur carrée.

Deux transformations non linéaires particulièrement intéressantes sont indiquées par ces résultats. D'abord, la transformation arctanh $(\sqrt{\hat{C}})$ qui possède la particularité que la variable de transformation est quasi Gaussienne, est le seul mode non linéaire qui ne varie pas en fonction de la valeur réelle C, selon l'ordre 1/N. Deuxièmement, la seule transformation non linéaire qui soit exempte de déviation, quelle que soit la valeur réelle de la grandeur cohérente, est log.$(\hat{C})$, selon l'ordre 1/N.

L'extension de ces résultats selon l'ordre $1/N^2$ est possible et permise pour des valeurs de N de relativement faible grandeur ; cependant, les relations sont un peu plus compliqués.

SUMMARY

Previous results on the statistics of the magnitude-squared-coherence estimate $\hat{C}$ and the magnitude-coherence estimate $\sqrt{\hat{C}}$ have required simplification of hypergeometric functions ${}_3F_2$, sometimes with great labor. Now this situation can be avoided by using relatively simple asymptotic relations that give, for example, the bias, variance, and mean-square-error of any nonlinear transformation of $\hat{C}$. These relations involve only N, the number of pieces of data averaged in the coherence estimate, and the nonlinear transformation and its first two derivatives, evaluated at true value C of the magnitude-squared coherence.

Two particularly attractive nonlinear transformations are indicated by these results. First, the arc tanh $\left(\sqrt{\hat{C}}\right)$ transformation, which possesses the feature that the transformed variable is nearly Gaussian, is the only nonlinear device that has variance independent of the true coherence value C to order 1/N. Second, the only nonlinear transformation that is unbiased, regardless of the true coherence value, is log $(\hat{C})$ to order 1/N.

Extensions of these results to order $1/N^2$ is possible and warranted for values of N that are not large; however, the relations are somewhat more complicated.

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INTRODUCTION

Expressions for the probability density function, the cumulative distribution function, and any moment of the estimates of magnitude-squared coherence (MSC) and magnitude coherence (MC) are available in references 1-5. The exact expressions for the moments usually involve a generalized hypergeometric function ${}_3F_2$ and require a time-consuming computer effort for their evaluation. Also, the fundamental dependence of statistics like the bias, variance, and mean-square error on the number of averages N and the true-coherence value are not obvious because of the lack of significant results for the ${}_3F_2$ function.

In reference 6, this shortcoming was partially alleviated by deriving asymptotic results for the MSC estimate and doing curve-fitting for the MC estimate, thereby obtaining relatively simple relations for the bias, variance, and mean-square error in these two cases. However, whenever a different nonlinear transformation of the MSC estimate is considered, the analytical effort must begin anew to determine the fundamental behavior of the statistics such as moments. For example, in reference 7, the nonlinear transformation arc tanh $\left(\sqrt{\hat{C}}\right)$ of MSC estimate $\hat{C}$ was shown to yield a nearly Gaussian random variable, thereby facilitating calculation of confidence limits for coherence detectors. However, the mean and variance of the nearly Gaussian random variable were deduced by a time-consuming trial-and-error curve-fitting procedure. We will rectify this situation by deriving simple asymptotic relations for large N for any nonlinear transformation. In this way we also deduce new transformations with particularly desirable behavior.

ESTIMATION OF MAGNITUDE-SQUARED COHERENCE

The complex coherence between two jointly stationary random processes x(t) and y(t) is defined as

$$\gamma_{xy}(f) = \frac{G_{xy}(f)}{\left[G_{xx}(f)\; G_{yy}(f)\right]^{1/2}} , \tag{1}$$

where

$G_{xy}(f)$ is the cross-spectral density at frequency f and

$G_{xx}(f)$ and $G_{yy}(f)$ are the autospectral densities.

The MSC is

$$C(f) = |\gamma_{xy}(f)|^2 . \tag{2}$$

Frequently the MSC is estimated according to

$$\hat{C}(f) \equiv \frac{|\hat{G}_{xy}(f)|^2}{\hat{G}_{xx}(f)\;\hat{G}_{yy}(f)} = \frac{\left|\sum\limits_{n=1}^{N} X_n(f)\, Y_n^*(f)\right|^2}{\sum\limits_{n=1}^{N} |X_n(f)|^2 \sum\limits_{n=1}^{N} |Y_n(f)|^2} , \tag{3}$$

where

N is the number of data segments employed and

$X_n(f)$, $Y_n(f)$ are the (discrete) Fourier transforms of the n-th weighted data segments x(t) and y(t); see reference 3 for example.

The statistics of a nonlinearly transformed version $g(\hat{C})$ of MSC estimate $\hat{C}$ are of interest here. We drop frequency dependence f for notational simplicity.

AVERAGE VALUE OF TRANSFORMED COHERENCE ESTIMATE

The probability density function of MSC estimate $\hat{C}$ is given by reference 1, eq. (2) et seq. as

$$p(x) = (N-1)\left(\frac{1-C}{1-Cx}\right)^2 \left(\frac{(1-C)\,(1-x)}{1-Cx}\right)^{N-2} P_{N-1}\left(\frac{1+Cx}{1-Cx}\right) \quad \text{for } 0 \le x \le 1 ,\; C < 1 , \tag{4}$$

where P_{N-1} is a Legendre polynomial. If $\hat{C}$ is subjected to nonlinear transformation $g(\hat{C})$, the average value of the output is

$$A = E\{g(\hat{C})\} = \int_0^1 dx\; p(x)\; g(x). \tag{5}$$

For large N, probability density p(x) is peaked about x = C; see reference 1, figures 1a-1h. Accordingly the major contribution to (5) will come from this neighborhood, so we expand transformation g about this point:

$$A = \int_0^1 dx\; p(x) \sum_{n=0}^{\infty} \frac{1}{n!}\, g^{(n)}(C)\; (x-C)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\, g^{(n)}(C)\; \nu_n , \tag{6}$$

where

$$\nu_n \equiv \int_0^1 dx\; p(x)\; (x-C)^n \tag{7}$$

is the n-th moment of $\hat{C}$ about C.

An expression for general moment $E\{\hat{C}^m\}$ is given in reference 6, page 2, along with specific simpler results for m = 1 and 2. If these latter results are expanded in a series in powers of N^{-1}, we find for (7),

$$\nu_1 = \frac{(1-C)^2}{N} + \frac{2C(1-C)^2}{N^2} + O(N^{-3}) ,$$
$$\nu_2 = \frac{2C(1-C)^2}{N} + \frac{2(1-C)^2\,(1-6C+7C^2)}{N^2} + O(N^{-3}) . \tag{8}$$

Furthermore, if the asymptotic behavior of the Legendre polynomial in (4) is employed (reference 8, page 194, eq. 8.21.1), we find, to highest order, that for C > 0 (reference 9)

$$\nu_n \sim \begin{cases} \dfrac{n!\; C^{n/2}(1-C)^n}{(n/2)!\; N^{n/2}} & \text{for n even} \\[2ex] \dfrac{n!\,C^{(n-1)/2}\,(1-C)^n\,[1+C+n(1-3C)]}{\left(\frac{n-1}{2}\right)!\; 2\; N^{(n+1)/2}} & \text{for n odd} \end{cases} \quad \text{as } N \to \infty \tag{9}$$

In particular, it is seen that ν_3 and ν_4 are $O(N^{-2})$, and ν_n for $n \ge 5$ is $O(N^{-3})$ or smaller.

Combining (8) and (9) in (6), we obtain average value

$$A = g(C) + \frac{(1-C)^2}{N}\,[g'(C) + Cg''(C)] + O(N^{-2}) \tag{10}$$

The term of $O(N^{-2})$ can be determined explicitly from (8) and (9); it involves $g^{(n)}(C)$ for n = 1,2,3,4 (see reference 9). An alternative approximation follows from (10) and is useful in some cases:

$$A \cong g\left(C + \frac{(1-C)^2}{N}\left[1 + C\,\frac{g''(C)}{g'(C)}\right]\right) . \quad (11)$$

Expressions (10) and (11) are the desired results; they give the average value of any nonlinearly transformed version of MSC estimate $\hat{C}$ to order N^{-1}. They are valid only in regions where nonlinear transformation g can be reasonably approximated by its first two derivatives in (6).

As an example, consider the transformation discussed in the introduction:

$$g(x) = \text{arc}\tanh(x^{1/2}),\; g'(x) = \frac{1}{2x^{1/2}(1-x)},\; g''(x) = \frac{3x-1}{4x^{3/2}(1-x)^2} \quad (12)$$

Substitution in (11) gives average value

$$A \cong \text{arc}\tanh\left(\left(C + \frac{1-C^2}{2N}\right)^{1/2}\right) , \quad (13)$$

in agreement to order N^{-1} with reference 7, eq. (8). The original form (10) is actually less useful near C = 0 since it tends to ∞ due to a term in $C^{-1/2}$.

The terms after g(C) in (10) represent the bias of the distorted MSC estimate. It can be seen that the dominant term of this bias is zero if, and only if, the nonlinear transformation is of the form

$$g(\hat{C}) = a \ln(\hat{C}) + b \quad (14)$$

That is, the logarithm of the MSC estimate is the only unbiased estimator to order N^{-1}. The correction terms of (13) and (14) to order N^{-2} are given in reference 9.

VARIANCE OF TRANSFORMED COHERENCE ESTIMATE

Since average value (10) holds for any nonlinear transformation, we can apply it to $g^2(\hat{C})$ as well as to $g(\hat{C})$; we then combine these results to find that the variance of transformed coherence $g(\hat{C})$ is given by the simple expression

$$V = \frac{2C(1-C)^2}{N}\,[g'(C)]^2 + O(N^{-2}) , \quad (15)$$

where $g''(C)$ drops out. This result corroborates reference 6, eqs. (9) and (33), for g(x) = x and $x^{1/2}$, respectively, i.e., the MSC and the MC estimates.

If we want the variance in (15) to be independent of true coherence value C, say equal to K/N, we can solve the differential equation and we find (see (12))

$$g(\hat{C}) = (2K)^{1/2}\,\text{arc}\tanh\left(\hat{C}^{1/2}\right) + b . \quad (16)$$

Thus, this nonlinearity is the only transformation of $\hat{C}$ that has variance independent of the actual value C to order N^{-1}. This behavior has already been noted in reference 7, eq. (9).

CONCLUSION

The fundamental behavior of a nonlinearly transformed coherence estimate can be deduced from fairly simple equations in terms of two derivatives of the transformation. These equations are asymptotic in powers of 1/N. Additional terms in the expression have been derived; their effects on the conclusions above are given in reference 9.

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