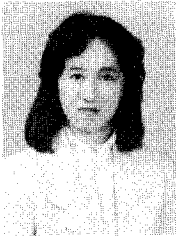


ε -separating nonlinear digital filters**Filtres numériques non linéaires ε -saturés****Kaoru ARAKAWA**

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SUMMARY

A new type of nonlinear digital filters which aims at removing small-amplitude random noises from a signal containing large-amplitude abrupt changes is proposed. These filters are constructed by nonlinearizing linear digital filters with a simple nonlinear function, and the difference between the input and the output of these filters is limited to a certain finite value. These filters are very easy to be realized and work quite effectively. We name these nonlinear digital filters as ε -separating nonlinear digital filters (ε -separating filters for short).

The linear filters used in the nonlinearization are usual linear digital filters used for noise reduction. First, ε -separating filters using a nonrecursive linear low-pass filter and a recursive one are proposed, and then an ε -separating filter using a Kalman filter is proposed. They are named as an ε -filter, a recursive ε -filter, and an ε -Kalman filter, respectively. The ε -Kalman filter can take the local statistical characteristics of the signal into consideration as well as the abrupt changes.

Moreover, a generalized form of the ε -separating filter is proposed.

Finally, computer simulations show the effectiveness of these ε -separating filters and their statistical characteristics are presented.

KEY WORDS

Nonlinear digital filter, noise reduction, low-pass filter, Kalman filter, signal with abrupt changes.

RÉSUMÉ

Nous proposons un nouveau type de filtres numériques extrayant un signal à variations brusques d'un bruit aléatoire d'amplitude limitée. Ces filtres sont obtenus en modifiant la sortie d'un filtre linéaire par une transformation non linéaire bornant supérieurement la différence entre la sortie et l'entrée. Ces filtres, faciles à réaliser, sont très efficaces. Nous proposons de les dénommer : filtres numériques non linéaires ϵ -saturés.

La partie linéaire est un des filtres usuellement employés pour la réduction de bruit. Nous envisageons successivement des filtres numériques non linéaires, non récursifs passe-bas, récursifs et de Kalman. Le filtre de Kalman est construit à partir d'un modèle statistique du signal. L'efficacité des filtres numériques non linéaires ϵ -saturés, non récursifs passe-bas, r de Kalman. Le filtre de Kalman est construit à partir d'un modèle statistique du signal. L'efficacité des filtres numériques non linéaires ϵ -saturés est illustrée sur des signaux simulés et leurs caractéristiques fonctionnelles moyennes sont présentées.

MOTS CLÉS

Filtre non linéaire, réduction de bruit, filtre passe-bas, filtre de Kalman, signal contenant des changements rapides.

1. Introduction

Linear filters have been used as the main method for noise reduction. However, we often encounter the case that the linear filtering is not effective. For example, when random noises are superimposed on image signals, the linear filters cannot remove the noises effectively, because the linear filters reduce the sharpness of the edges as well as the noises.

Therefore, nonlinear filters are required for such noise reduction, that is, reducing noises superimposed on signals which contain large-amplitude abrupt changes, such as edges of images, without degrading the abrupt changes. Several nonlinear filters have been proposed for this purpose. Moore *et al.* proposed an E-filter which transform the time axis nonlinearly [1, 2], and Tukey proposed a filter which uses running medians [3, 4].

In this paper, a new type of nonlinear digital filter is proposed. This filter aims at reducing small-amplitude random noises superimposed on signals containing large-amplitude abrupt changes. This filter is simply realized by combining usual linear digital filter and a nonlinear function so that the difference between the input and the output is limited to a certain finite value ϵ . We name this filter as an ϵ -separating filter. Compared with the other nonlinear digital filters, this ϵ -separating filter can be easily realized and can reduce small-amplitude random noises quite effectively.

Various kinds of ϵ -separating filters can be considered depending on the original linear digital filters. In this paper, ϵ -separating filters using linear low-pass filters (nonrecursive and recursive) and that using a Kalman filter are proposed. These filters are named as an ϵ -filter, a recursive ϵ -filter, and an ϵ -Kalman filter, respectively. The ϵ -filter is the simplest of them, while the ϵ -Kalman filter has the ability to consider the statistical characteristics of the input signal as well as the abrupt changes.

2. The fundamental idea of ϵ -separating nonlinear digital filter

Suppose that an input signal contains large-amplitude abrupt changes, and that small-amplitude random noises whose amplitude is less than a certain finite value are superimposed on the signal as shown in Figure 1 a. If the input is processed by a usual linear-low-pass filter for the noise reduction, the sharpness of the abrupt changes is also reduced. In order to reduce the noise and to preserve the sharpness of the abrupt changes, the difference between the input and the output must be limited to a certain finite value such as ϵ as shown in Figure 1 b. Therefore, we consider a nonlinear digital filter that corresponds to a function that limits the difference between the input and the output to a certain finite value ϵ . This is the fundamental idea of the ϵ -separating nonlinear digital filter (ϵ -separating filter for short).

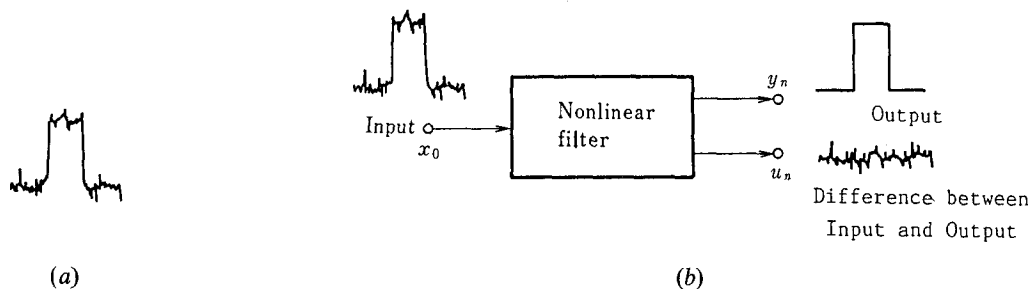


Fig. 1. — Function of the ϵ -separating filter. (a) An example of an input signal. (b) Block diagram of the ϵ -separating filter.

3. ϵ -separating filters using linear low-pass filters

3.1. PRINCIPLE OF ϵ -SEPARATING FILTERS USING LINEAR LOW-PASS FILTERS

Generally, a linear low-pass filter is expressed as follows.

$$(1) \quad y_n = \sum_{k=-N}^M a_k x_{n-k} + \sum_{l=1}^L b_l y_{n-l}$$

Here, x_m and y_m are the input and the output at time m respectively, and a_k and b_l are the coefficients of the low-pass filter.

In order to keep a DC level unchanged, the coefficients a_k and b_l satisfy the following equation.

$$(2) \quad \sum_{k=-N}^M a_k + \sum_{l=1}^L b_l = 1.$$

Using this equation, equation (1) can be transformed as follows.

$$(3) \quad y_n = \sum_{k=-N}^M a_k \{x_{n-n_0} - (x_{n-n_0} - x_{n-k})\} + \sum_{l=1}^L b_l \{x_{n-n_0} - (x_{n-n_0} - y_{n-l})\} \\ = x_{n-n_0} - \sum_{k=-N}^M a_k (x_{n-n_0} - x_{n-k}) + \sum_{l=1}^L b_l (x_{n-n_0} - y_{n-l}).$$

Suppose that nonlinear functions $F_1(\cdot)$ and $F_2(\cdot)$, as shown in Figure 2, satisfying

$$(4) \quad |F_1(\cdot)| \leq \epsilon_1, \quad |F_2(\cdot)| \leq \epsilon_2$$

are used to modify equation (3) as follows,

$$(5) \quad y_n = x_{n-n_0} - \sum_{k=-N}^M a_k F_1(x_{n-n_0} - x_{n-k}) + \sum_{l=1}^L b_l F_2(x_{n-n_0} - y_{n-l});$$

the difference between the input and the output of the filter expressed as equation (5) is limited to a finite value as follows.

$$(6) \quad |x_{n-n_0} - y_n| \leq \sum_{k=-N}^M |a_k| \cdot |F_1(x_{n-n_0} - x_{n-k})| + \sum_{l=1}^L |b_l| \cdot |F_2(x_{n-n_0} - y_{n-l})| \\ \leq \epsilon_1 \sum_{k=-N}^M |a_k| + \epsilon_2 \sum_{l=1}^L |b_l| \equiv \epsilon.$$

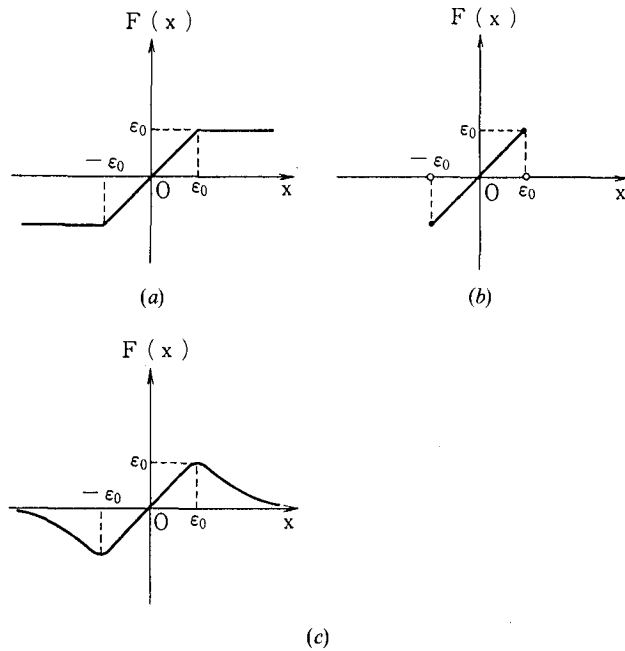


Fig. 2. — Examples of nonlinear function $F(\cdot)$ [ϵ_0 corresponds to ϵ_1 for $F_1(\cdot)$, and ϵ_2 for $F_2(\cdot)$].

Thus, this filter satisfies the condition for the ϵ -separating filter. Here, the time delay between the input and the output is presented by n_0 . The schematic diagram of this filter is shown in Figure 3 [5].

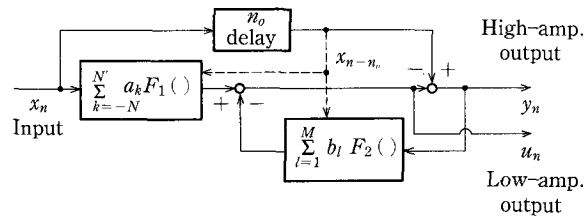


Fig. 3. — Schematic diagram of the ϵ -separating filter using a low-pass filter.

When the original low-pass filter is nonrecursive, the ϵ -separating filter is also nonrecursive and is expressed as

$$(7) \quad y_n = x_{n-n_0} - \sum_{k=-N}^M a_k F(x_{n-n_0} - x_{n-k})$$

We call this filter as an ϵ -filter. Contrarily, when $b_l \neq 0$, we call the ϵ -separating filter as a recursive ϵ -filter. In the case of the ϵ -filter, if $a_k = a_{-k}$, the delay between the input and the output does not exist. Therefore, x_{n-n_0} is equal to x_n .

3.2. HOW THE ϵ -SEPARATING FILTER SMOOTHS SIGNALS

Equation (5) can be expressed as follows from equation (1).

$$(8) \quad y_n = \sum_{k=-N}^M a_k x'_{n-k} - \sum_{l=1}^L b_l y'_{n-l}$$

When $F(\cdot)$ as shown in Figure 2b is adopted as $F_1(\cdot)$ and $F_2(\cdot)$, x'_{n-k} and y'_{n-l} are given by

$$(9) \begin{cases} x'_{n-k} = \begin{cases} x_{n-k}, & \text{if } |x_{n-n_0} - x_{n-k}| \leq \epsilon_1; \\ x_{n-n_0}, & \text{if } |x_{n-n_0} - x_{n-k}| > \epsilon_1; \end{cases} \\ y'_{n-l} = \begin{cases} y_{n-l}, & \text{if } |x_{n-n_0} - y_{n-l}| \leq \epsilon_2; \\ x_{n-n_0}, & \text{if } |x_{n-n_0} - y_{n-l}| > \epsilon_2. \end{cases} \end{cases}$$

This filter differs from a linear filter in using x_{n-n_0} instead of x_{n-k} when $|x_{n-n_0} - x_{n-k}|$ is greater than ϵ_1 and using x_{n-n_0} instead of y_{n-l} when $|x_{n-n_0} - y_{n-l}|$ is greater than ϵ_2 .

As an example, suppose that the input signal x_n is obtained as shown in Figure 4a. When the input is processed with the filter expressed as equation (5) and x_{n-n_0} is located at the point A, the signal is smoothed by replacing all the input signal points in the filter window beyond ϵ_1 of the point A with the value of the point A as shown in Figure 4b. Therefore, the point A is smoothed without being affected by the abrupt change. In this case y'_{n-l} is equal to y_{n-l} because $|y_{n-l} - x_{n-n_0}| \leq \epsilon_2$ for all. Similarly, when x_{n-n_0} is at the point B, the signal is smoothed by transforming the input and the past output signal sequences to flat ones as shown in Figure 4c.

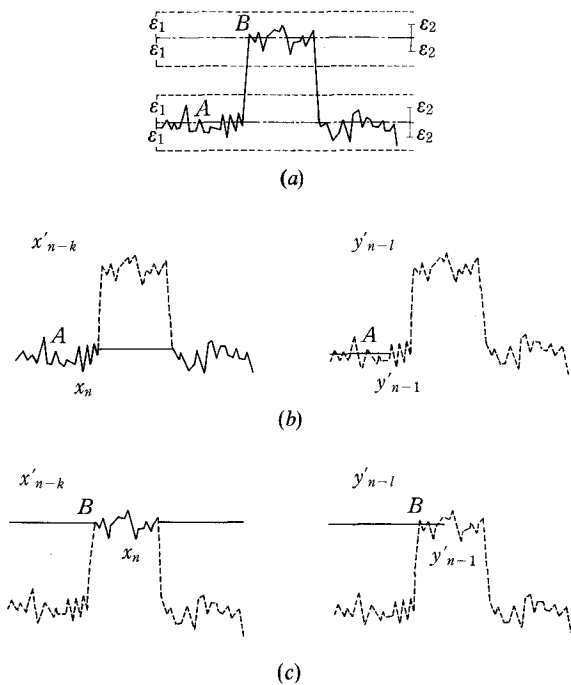


Fig. 4. — How the ε-separating filter smooths a signal.

In this way, the filter expressed as equation (5) can reduce small-amplitude random noises, preserving the sharpness of the abrupt changes.

Even if other types of nonlinear functions as shown in Figure 2 are adopted as $F_1(\cdot)$ and $F_2(\cdot)$, the ability for smoothing is similarly explained.

The value of ϵ_1 should be set to the value of double the maximum amplitude of noises and that of ϵ_2 the half of that in order to filter out the noises adequately. If the noise is expressed as a statistical process such as gaussian, the noise does not possess the maximum

amplitude. However, the amplitude of the noises we actually obtain is usually limited to a certain finite value. For example, a gaussian noise can be considered as a truncated-gaussian noise in actual cases. Therefore, we can usually find the maximum value of the noises. Some noises may exceed the value, however, such a case is very rare and we can neglect it.

3.3. SOME CHARACTERISTICS OF THE RECURSIVE ε-FILTER

The recursive ε-filter is more complicated than the non-recursive one, because the value of the delay and the problem of stability have to be considered. The delay n_0 is set to the value of the delay of the original linear recursive low-pass filter. However, a recursive low-pass filter usually has a nonlinear phase as shown in Figure 5a. Therefore, n_0 should be set to the value which is obtained by fitting a straight line to the phase characteristics curve in the pass-band as is shown in Figure 5b; n_0 is the approximate integer of the gradient of the straight line.

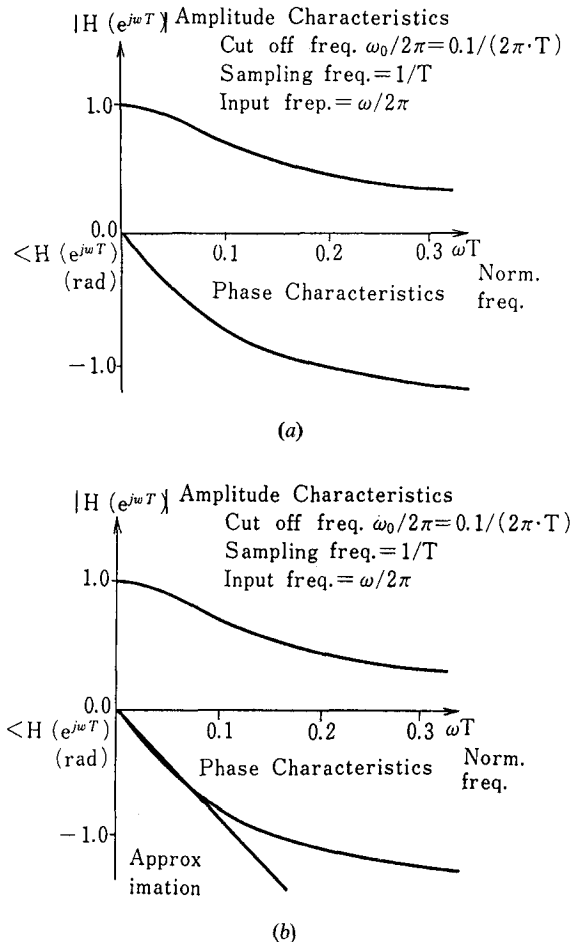


Fig. 5. — Amplitude and phase characteristics of a recursive low-pass filter. (a) Characteristics of a recursive low-pass filter. (b) Approximating straight line of the phase characteristics.

Next, as to the stability problem, the recursive ε-filter is BIBO stable (the output is bounded if the input is bounded) as shown in equation (6). However, large-amplitude oscillations transiently occur in the output after a large-amplitude change, when order of the

original recursive filter, that is L , is greater than two; when L is large, ε in equation (6) can be a great value. This oscillation occurs because of the artificial transformation of the output sequence y_{n-1} . This phenomenon corresponds to a limit-cycle (see Appendix).

In order to avoid this oscillation, the recursive ε -filter has to be constructed in a cascade form of the recursive ε -filters in which L is less than or equal to two. When $L=2$, such an oscillation occurs if the cut-off frequency of the original low-pass filter is too low, however, usually such an oscillation is negligible.

4. ε -Kalman filter

In this section, an ε -separating filter which is based on a Kalman filter is proposed.

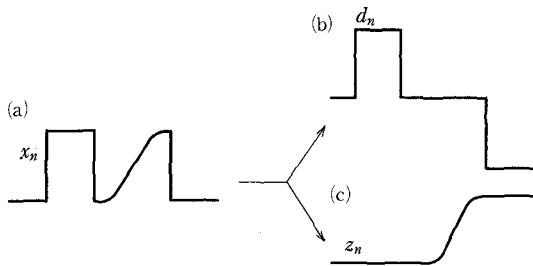


Fig. 6. - An example of separating a compound signal.

Generally, a signal x_n which contains abrupt changes and continuous changes as shown in Figure 6a can be expressed as a summation of a step-like signal d_n and a continuous signal z_n as shown in Figure 6b and c. Moreover, z_n is expressed as follows.

$$(10) \quad z_n = A_{n-1} z_{n-1} + b_{n-1} u_{n-1},$$

Here z_n denotes a vector $(z_n, z_{n-1}, \dots, z_{n-N+1})^T$, u_{n-1} a vector $(u_{n-1}, 0, \dots, 0)^T$ where u_{n-1} denotes a sampled white gaussian noise with zero mean, b_{n-1} a coefficient, and A_{n-1} a matrix as follows.

$$(11) \quad A_{n-1} = \begin{pmatrix} a_1^{(n-1)} & \dots & a_{N-1}^{(n-1)} & a_N^{(n-1)} \\ 1 & & 0 & 0 \\ & & & 0 \\ \dots & 1 & 0 & \dots \end{pmatrix}$$

Next, suppose that a white noise is superimposed on the signal x_n and an observation of the signal is obtained as $y_n = x_n + w_n$, where w_n is a white noise whose average is equal to zero and variance is W .

Now we want to estimate the value of x_n from the observations y_n, y_{n-1}, \dots, y_0 . Let us consider a filter which estimates the value of d_n as \hat{d}_n and subtracts it from the observation y_n as shown in Figure 7 and then processes the remaining signal (Fig. 7c) with a linear Kalman filter. Such a filter can be realized as follows. First of all, the predicted value of x_n, z_n and d_n at time $(n-1)$ are expressed as \tilde{x}_n, \tilde{z}_n and \tilde{d}_n as follows.

$$(12) \quad \tilde{x}_n = \tilde{z}_n + \tilde{d}_n = c \tilde{z}_n + \tilde{d}_n,$$

$$(13) \quad \tilde{z}_n = A_{n-1} \hat{z}_{n-1},$$

$$(14) \quad \tilde{d}_n = \hat{d}_{n-1}.$$

Here, c denotes a vector $(10 \dots 0)$, and \hat{z}_{n-1} and \hat{d}_{n-1} are the estimated value of z_{n-1} and d_{n-1} respectively at time $(n-1)$.

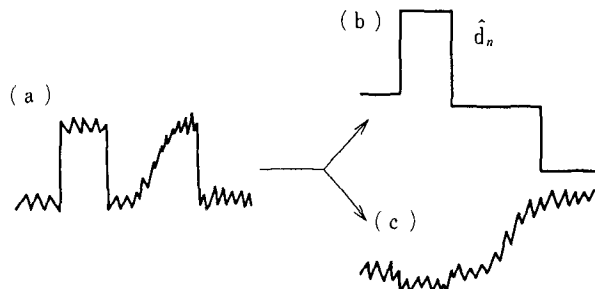


Fig. 7. - An example of separating a noisy compound signal.

The estimated value of d_n , that is \hat{d}_n , can be obtained as follows.

$$(15) \quad \hat{d}_n = \tilde{d}_n + (y_n - \tilde{x}_n) - F(y_n - \tilde{x}_n) \\ = y_n - \tilde{z}_n - F(y_n - (\tilde{z}_n + \tilde{d}_n)).$$

Here, $F(\cdot)$ is a nonlinear function as shown in Figure 2. Equation (15) implies that when the amplitude of the prediction error is smaller than ε_0 , that is $|y_n - \tilde{x}_n| \leq \varepsilon_0$, the input signal is considered to be continuous and \hat{d}_n keeps to take the value of \hat{d}_{n-1} . On the other hand, when $|y_n - \tilde{x}_n| > \varepsilon_0$, the input is considered to possess an abrupt change and then \hat{d}_n takes a value which is close to the sum of \hat{d}_{n-1} and the prediction error.

The estimated value of z_n is obtained by the algorithm of a Kalman filter as follows, where $y_n - \hat{d}_n$ corresponds to the input of the Kalman filter.

$$(16) \quad \hat{z}_n = \tilde{z}_n + P_n c^T W^{-1} [(y_n - \hat{d}_n) - \tilde{z}_n] \\ = \tilde{z}_n + P_n c^T W^{-1} F(y_n - (\tilde{z}_n + \tilde{d}_n)).$$

Here, P_n denotes a matrix $E[(z_n - \hat{z}_n)(z_n - \hat{z}_n)^T]$ obtained as.

$$(17) \quad P_n = (M_n^{-1} + c^T W^{-1} c)^{-1},$$

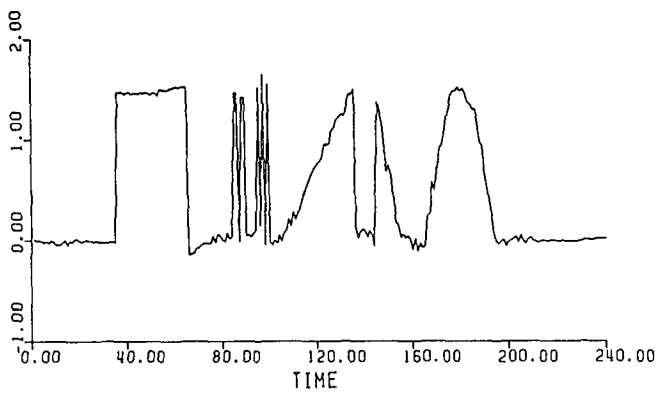
where M_n is calculated as

$$(18) \quad M_n = A_{n-1} P_{n-1} A_{n-1}^T + b_{n-1}^2 U_{n-1}.$$

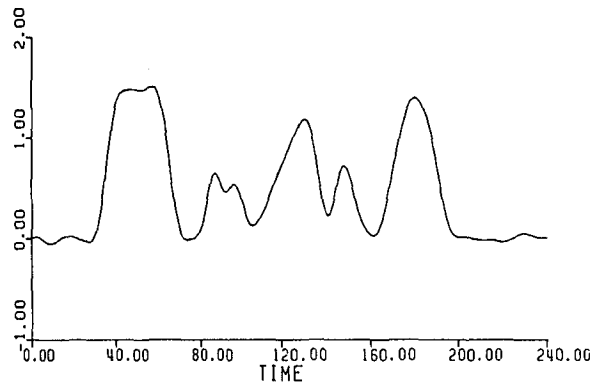
Here U_{n-1} denotes $E[u_{n-1} u_{n-1}^T]$.

Finally, the estimated value of x_n , that is \hat{x}_n , is obtained as follows from equations (15) and (16)

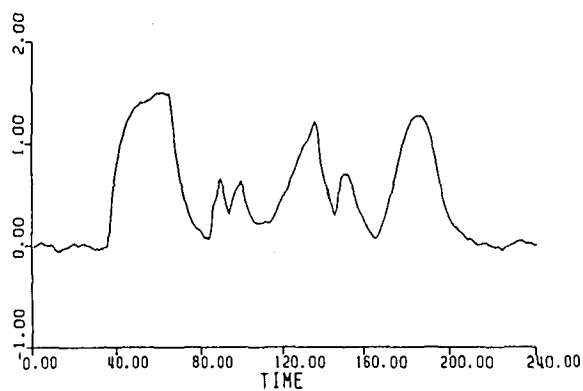
$$(19) \quad \hat{x}_n = c \hat{z}_n + \hat{d}_n \\ = y_n - (1 - P_n(1,1)W^{-1}) F(y_n - \tilde{x}_n).$$



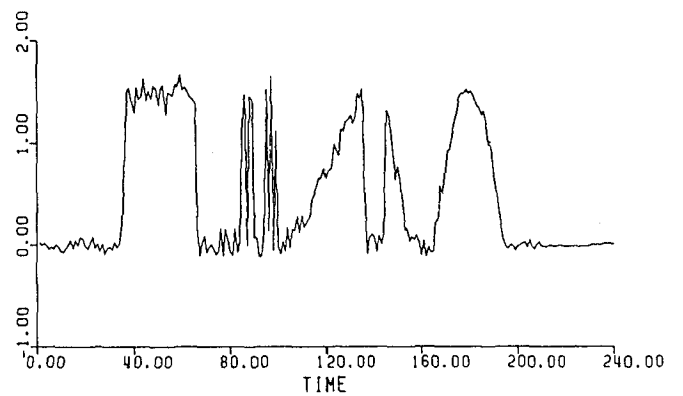
(e)



(f)



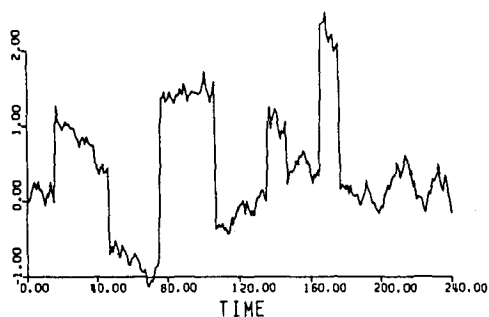
(g)



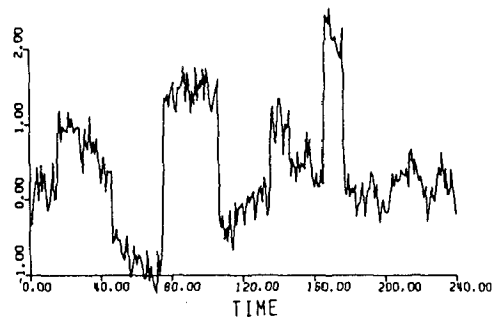
(h)

Fig. 9. — Results of processing a test signal with the ϵ -separating filters. (a) Original signal. (b) Input signal. [(a) + white gaussian noise]. (c) Output of the ϵ -filter. (d) Output of the recursive ϵ -

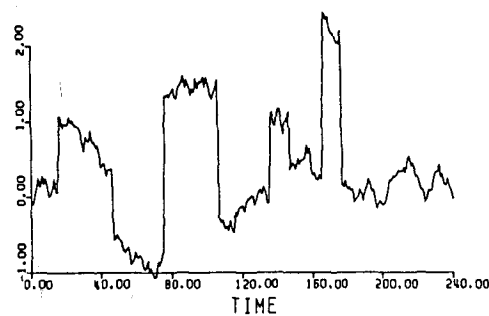
filter. (e) Output of the ϵ -Kalman filter. (f) Output of a linear nonrecursive low-pass filter. (g) Output of a linear recursive low-pass filter. (h) Output of a linear Kalman filter.



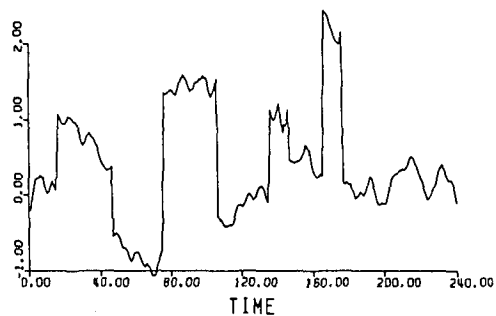
(a)



(b)



(c)



(d)

Fig. 10. — Results of processing a test signal expressed by a statistic model with the ϵ -separating filters. (a) Original signal. (b) Input

signal [(a) + white gaussian noise]. (c) Output of the ϵ -Kalman filter (MSE=0.0067). (d) Output of the ϵ -filter (MSE=0.0068).

5.2. RESULTS OF PROCESSING A TEST SIGNAL EXPRESSED BY A STATISTICAL MODEL

Figure 10a shows an original test signal which is expressed as a sum of a step-like signal and a first-order Markov process z_n as

$$(22) \quad z_n = z_{n-1} + u_{n-1},$$

where u_{n-1} is a white gaussian noise whose variance is 0.01. Figure 10b shows the input signal obtained as a sum of the original test signal and a white gaussian noise whose variance is 0.0225. The result of processing this input with an ε-Kalman filter is shown in Figure 10c, and that with an ε-filter is in Figure 10d. These figures show that the ε-Kalman filter reduces small-amplitude white noises, while the ε-filter reduces small-amplitude high-frequency noises.

5.3. STATISTICAL CHARACTERISTICS OF THE ε-SEPARATING FILTER

We show the statistical characteristics of the ε-separating filter using a certain input signal model.

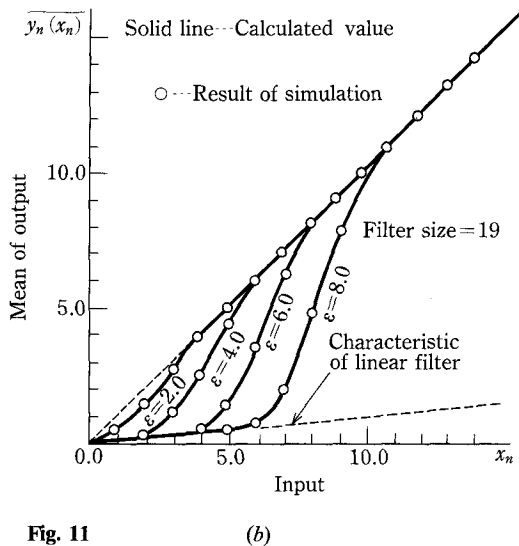
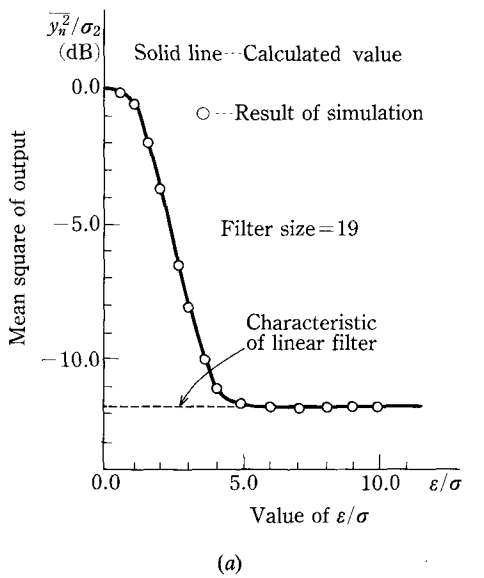


Fig. 11

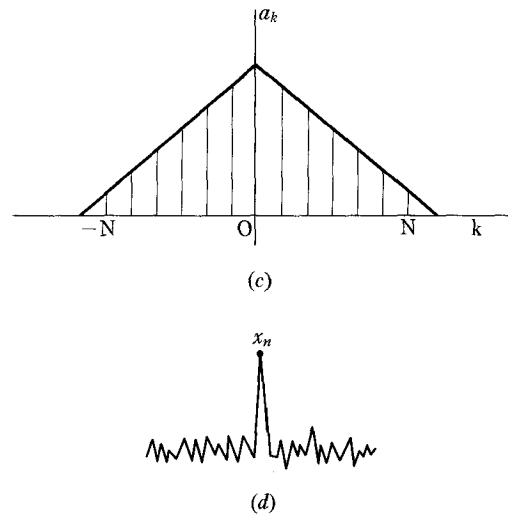


Fig. 11. — Statistical characteristics of the ε-filter. (a) Mean square square of the output of the ε-filter versus ε/σ. The input is a white gaussian noise. (b) Mean of the output of the ε-filter versus x_n . The input is an impulsive signal. (c) Filter coefficients a_k of the ε-filter. (d) The input impulsive signal.

Figure 11a presents the mean square of the output of the ε-filter for various values of ε/σ, when the input is a white gaussian noise whose average is zero and variance is σ^2 . The filter coefficients a_k take the shape as shown in Figure 11c where $N=9$. Figure 11a shows that when ε is too small, the input is not smoothed at all, and when ε is large enough, this filter works as a linear low-pass filter. We can see that the value of ε should be set to at least five times of the standard deviation of the noise in order to reduce them adequately.

Figure 11b presents the mean of the output of the ε-filter, for various values of x_n , when the input is an impulsive signal degraded by white gaussian noises as shown in Figure 11d. The filter coefficients are the same as those used in Figure 11a, and the standard deviation of the noises is fixed to 1.0. The amplitude of the impulsive signal x_n is changed, and for each value of x_n , the average of the output y_n is calculated and plotted in this figure. Here, $n_0=0$ and x_{n-n_0} in equation (5) corresponds to x_n . This figure shows that when the impulsive signal is too small, it is reduced as a noise, and when the impulsive signal is large enough, it can be preserved in the output. In order to preserve a small impulsive signal, ε should be as small as possible.

As to the other types of ε-separating filters, similar characteristics can be obtained.

6. Conclusion

In this paper, a new type of nonlinear digital filter named as an ε-separating filter is proposed. These filters are realized by nonlinearizing ordinary linear digital filters so that the difference between the input and the output can be limited to a certain finite value ε. These nonlinear digital filters can effectively reduce small-amplitude random noises superimposed

TABLE
Comparison of ϵ -separating filters.
(0... Positive)

	Consideration of the statistic characteristics of signals	Easiness of realization	Removal of high-frequency noise	Removal of white noise
ϵ -filter		0	0	
Recursive ϵ -filter			0	
ϵ -Kalman filter	0			0

on a signal which contain large-amplitude abrupt changes.

Three kinds of ϵ -separating filters are proposed, that is, an ϵ -filter, a recursive ϵ -filter and an ϵ -Kalman filter. The comparison of these filters is presented in Table.

Computer simulations show that these filters are quite effective.

As to the shape of the nonlinear function in the ϵ -separating filters, the shape which experimentally gives preferable results is used in the computer simulations of this paper. For example, in the case of the ϵ -filter, the nonlinear function as shown in Figure 2b is used, and in the case of the recursive ϵ -filter, that as shown in Figure 2c is used. How to determine the optimum shape of the nonlinear function will be addressed in a later paper.

Appendix

Suppose an input signal as shown in Figure 12a is obtained. The output of the recursive ϵ -filter is calculated in the form of a linear filter as equation (8). The value of the past output y_{n-l} is replaced by y'_{n-l} in the following way.

- (1) When $n < m$, all the y_{n-l} 's ($1 \leq l \leq M$) satisfy $|x_{n-n_0} - y_{n-l}| \leq \epsilon_2$. Therefore, $y'_{n-l} = y_{n-l}$, and the recursive ϵ -filter works as a linear filter for the feedback path.
- (2) When $n = m$, all the y_{n-l} 's ($1 \leq l \leq M$) satisfy $|x_{n-n_0} - y_{n-l}| > \epsilon_2$. Therefore, $y'_{n-l} = s_{n-l}^{(m)} \approx x_{m-n_0}$ as shown in Figure 12b.
- (3) When $n = m + 1$, y_{n-1} satisfies $|x_{n-n_0} - y_{n-1}| \leq \epsilon_2$, but the other y_{n-l} 's do not. Therefore, $y'_{n-1} = y_{n-1}$ and $y_{n-l} = s_{n-l}^{(m+1)} \approx x_{m+1-n_0}$ ($2 \leq l \leq M$) as shown in Figure 12b. If $s_{m-l}^{(m)} = s_{m-l}^{(m+1)}$, this filter works as a linear recursive low-pass filter for the feedback path at the time point $m + 1$. However, since $s_{m-l}^{(m)}$ is not equal to $s_{m-l}^{(m+1)}$ in this case, the difference $s_{m-l}^{(m+1)} - s_{m-l}^{(m)}$ must be added to the feed-back path in expressing the performance of the recursive ϵ -filter as that of a linear recursive filter.

Also, when $n > m + 1$, y'_{n-l} is modified by adding such a small value.

In this way, some signal component, although it is small, is added to the feed-back path of the original linear recursive low-pass filter by transforming the

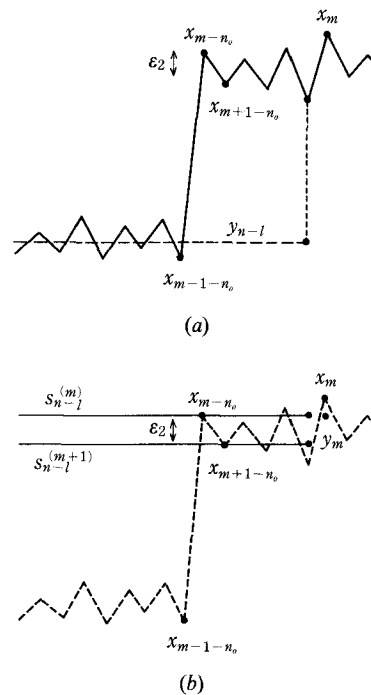


Fig. 12. — An example of the input signal which causes an oscillation in the recursive ϵ -filter. (a) An input signal sequence. (b) The transformed past output signal sequence.

past output signal sequence into a flat shape. Such a signal component corresponds to a quantization error in the feed-back path of a recursive digital filter, and accordingly causes an oscillation like a limit-cycle [7, 9].

Manuscrit reçu le 7 octobre 1986.

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