Resonance scattering theory: spherical and cylindrical cavities and inclusions

Théorie de la diffusion résonnante : cavités et inclusions sphériques et cylindriques

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Elastic waves: First development of theory of resonance scattering by cavities and inclusions in solids.

Electromagnetic waves: Establishment of the relation between resonance scattering theory and the Singularity Expansion Method (SEM). Theory of radio-wave propagation in the earth-ionosphere waveguide.

Nuclear theory: Basic development of coherent bremsstrahlung theory (editor of recent book). Development of a model of collective nuclear multipole vibrations, with application to electron scattering (author of two books). Photo-pion reactions. Collective model of ion-ion scattering. Neutrino reactions (editor of book).

SUMMARY

We consider the scattering of elastic dilatational and shear waves from cylindrical and spherical cavities and inclusions in an elastic medium. The normal mode series of the scattering amplitude is reformulated in terms of the S-function, and the poles of the S-function in the complex frequency plane are identified. The amplitude is rewritten as a "background term" including specular reflections and external surface waves, plus a series of (internal) resonance terms. This formulation is termed the "Resonance Scattering Theory" (RST). The connection between the resonances and the surface waves is established via expressing the complex-frequency poles of the scattering amplitude by the Regge poles in the complex-mode number plane, and the frequency resonances in successive modes are recognized as the Regge recurrences of surface-wave resonances. This permits us to obtain the dispersion curves of phase and group velocities of the (internal) surface waves from the eigenfrequencies of the cavity. We also mention experiments on ultrasonic scattering from cavities and inclusions.

KEY WORDS

Cavities, inclusions, Resonance scattering theory, RST, surface waves, complex-frequency poles, Regge poles, dispersion, phase velocity, group velocity.

RÉSUMÉ

Nous étudions la diffusion d'ondes élastiques de type dilatation ou cisaillement par des cavités et inclusions cylindriques ou sphériques. La série des modes normaux donnant l'amplitude de diffusion est récomposée en utilisant la fonction S, et les pôles de la fonction S dans le plan complexe des fréquences sont identifiés. L'amplitude est réécrite sous la forme d'un « terme de fond », comprenant réflexions spéculaires et ondes de surface externes, plus une série de termes de résonances (internes). Ce formalisme est appelé « Théorie de diffusion résonnante » [« Resonance Scattering Theory » (RST)]. Le rapport entre les résonances et les ondes de surface est établi en exprimant les pôles en fréquence complexe de l'amplitude de diffusion par les pôles de Regge dans le plan complexe du numéro de mode, et les résonances en fréquence dans les modes successifs sont reconnues comme les récurrences des résonances d'ondes de surface. Cela nous permet d'obtenir les courbes de dispersion pour la vitesse de phase et de groupe des ondes (internes) de surface, en utilisant les fréquences propres de la cavité. Nous mentionnons également des expériences sur la diffusion ultrasonore par des cavités et inclusions.

MOTS CLÉS

Cavités, inclusions, théorie de la diffusion résonnante, RST, ondes de surface, pôles de fréquence complexe, pôles de Regge, dispersion, vitesse de phase, vitesse de groupe.

CONTENTS

1. Introduction

The Resonance Scattering Theory (RST) has first been formulated ([1]; *see* also [2, 3]) for the case of sound scattering from a solid cylinder or sphere in a fluid. The resonance frequencies, which may e. g. be represented by a "pole diagram" in the complex frequency plane [4, 5], were shown to originate from the phase matching of surface waves [2, 3, 6]. When the frequency poles are transferred to the complex mode number plane [6], where they constitute the "Regge poles", they determine by their real and imaginary parts the propagation constant and attenuation of the surface waves, respectively; the dispersion and attenuation curves of the surface waves can be obtained from the resonance frequencies without having to go through the (complex) Watson transformation.

A similar program may be carried out for ultrasonic wave scattering from cavities of cylindrical [7-9] or spherical [10-13] shape, or from solid inclusions [14, 15]. This topic is chosen here to present the resonance theory, since surface waves and resonances in acoustic scattering from elastic objects are treated elsewhere in this issue [16, 17]. While experiments on resonant acoustic scattering from solids are quite numerous by now (*see*, e. g., [18, 19] and references quoted therein), ultrasonic experiments on scattering from cavities or inclusions as done so far [20-23] have not been specifically designed to detect resonances, so that resonance effects appear in their results only indirectly.

2. Resonance theory

We consider an infinite-cylindrical cavity in a solid (density ρ, wave speeds c_p, c_s) filled with a fluid (density ρ_f, sound speed c_f). The displacement field $\mathbf{u}$ in the solid is represented by a scalar (Φ) and a vector (Ψ) potential:

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi \tag{2.1}$$

(with $\nabla . \Psi = 0$), and a normally incident p-wave (suppressing exp-$i\omega t$) is given by:

$$\Phi_{\text{inc}} = e^{i(k_p x)} = \sum_{n=0}^{\infty} (2-\delta_{n0})\, i^n \mathrm{J}_n(k_p r) \cos n\varphi, \tag{2.2}$$

where $k_p = \omega/c_p$. Scattered elastic waves are of both p-type,

$$\Phi_{\text{sc}} = \frac{1}{2}\sum_{n=0}^{\infty} (2-\delta_{n0}) \times i^n (\mathrm{S}^{pp}-1)\, \mathrm{H}_n^{(1)}(k_p r) \cos n\varphi, \tag{2.3}$$

or s-type ("mode conversion"),

$$\Psi_{\text{sc}} = \frac{1}{2}\sum_{n=0}^{\infty} (2-\delta_{n0})\, i^n \mathrm{S}_n^{ps} \mathrm{H}_n^{(1)}(k_s r) \sin n\varphi, \tag{2.4}$$

where $k_s = \omega/c_s$. A standing wave exists in the fluid filler:

$$(2.5) \quad \Phi_f = \frac{1}{2} \sum_{n=0}^{\infty} (2-\delta_{n0})\, i^n C_n^p J_n(k_f r) \cos n\varphi,$$

where $k_f = \omega/c_f$. Similar equations hold [8] for s-wave incidence (assumed SV, so that only $\Psi \equiv \Psi_z \neq 0$), with coefficients S_n^{sp}, S_n^{ss} and C_n^s. The sets of S_n^{ik} form the 2×2 S-matrix.

Satisfying the boundary conditions (continuity of normal stresses and displacements, zero tangential stress) at the cylinder radius $r=a$ leads to 3×3 linear systems:

$$(2.6a) \qquad D_n^p S_n^p = E_n^p, \qquad D_n^s S_n^s = E_n^s,$$

where $D_n^{p,s}$ are 3×3 matrices and $E_n^{p,s}$ vectors, all with known elements [8] which consist of cylinder functions of order n and arguments $\alpha \equiv k_p a$, $\beta \equiv k_s a$ and $\gamma \equiv k_f a$, and:

$$(2.6b) \quad S_n^p = \begin{pmatrix} S_n^{pp} \\ S_n^{ps} \\ C_n^p \end{pmatrix}, \qquad S_n^s = \begin{pmatrix} -S_n^{sp} \\ S_n^{ss} \\ C_n^s \end{pmatrix}.$$

Using Cramer's rule, the elements S_n^{ik} are thus fractions with $D_n(\omega) \equiv \det(D_n^p) \equiv -\det(D_n^s)$ in the denominator. Resonances occur when $D_n(\omega)=0$; the complex solutions ω_{nj} $(j=1, 2, \ldots)$ represent resonance frequencies common to all four amplitudes S_n^{ik}. Alternately, at a given ω (real), $D_n(\omega)=0$ is an equation for n with complex solutions $n = \nu_j(\omega)$. While ω_{nj} are complex-frequency poles ("SEM" poles", *see* [17]) of the scattering amplitude, $\nu_j(\omega)$ are its complex mode number ("Regge") poles [16]; both are related through $D_\nu(\omega)=0$.

More explicitly, one may write for S_n^{ik}, e. g.,

$$(2.7a) \qquad S_n^{pp} = S_n^{(0)\,pp} \frac{Z_n^{(2)} - K_n}{Z_n^{(1)} - K_n},$$

where:

$$(2.7b) \qquad K_n = \frac{\rho}{\rho_f} \frac{\gamma J_n'(\gamma)}{J_n(\gamma)},$$

with $Z_n^{(i)}$ expressed by the elements of D_n^p, and $S_n^{(0)\,pp} \equiv \exp 2i\delta_n$ the value of S_n^{pp} for an empty cavity $(\rho_f = 0)$. Expanding $K_n(\alpha)$ about the resonance value $\alpha_{nj} = (\mathrm{Re}\,\omega_{nj})\, a/c_p$ leads to:

$$(2.8) \quad S_n^{pp} - 1 \cong e^{2i\delta_n} \times \left\{ \sum_{j=1}^{\infty} \frac{M_{nj}}{\alpha - \alpha_{nj} + (i/2)\Gamma_{nj}} + 2i\, e^{-i\delta_n} \sin \delta_n \right\},$$

where:

$$(2.9a) \qquad M_{nj} = (Z_n^{(1)} - Z_n^{(2)})/K_n'(a_{nj}),$$

$$(2.9b) \qquad \Gamma_{nj} = -2\, \mathrm{Im}\, Z_n^{(1)}/K_n'(\alpha_{nj}).$$

This is the Breit-Wigner form of the scattering amplitude, as developed in nuclear scattering [24]. It consists of a sum of resonant terms (with M_{nj}: internal resonances), and a smooth background term corresponding to the scattering from an empty cavity, second term in equation (2.8). However, this background still contains resonances corresponding to external (Franz) waves, as well as geometrical (specularly reflected, and transmitted) waves [16].

The quantities $\nu_j(\omega)$ are the propagation constants of surface waves, as in the Watson-transform method [16]. This can be seen, e. g., by expressing the resonance denominator in α in equation (2.8) by one linear in n (assumed a continuous variable) [6, 25]; this furnishes the Watson results without having to go through the complex arithethic. The surface wave parameters are thus:

$$(2.10a) \qquad c_j^{ph} = a\,\omega/\mathrm{Re}\,\nu_j(\omega)$$

(phase velocity),

$$(2.10b) \qquad c_j^{gp}(\omega) = a/\mathrm{Re}(d\nu_j/d\omega)$$

(group velocity), and (amplitude $\exp - \varphi/\varphi_l$):

$$(2.10c) \qquad \varphi_l = 1/\mathrm{Im}\,\nu_l$$

(attenuation). Their wavelength (in the fluid) is:

$$(2.10d) \qquad \lambda_j = \frac{2\pi a}{\mathrm{Re}\,\nu_j},$$

showing that a resonance ($\mathrm{Re}\,\nu_j \to n$) occurs when the circumference of the cavity equals an integral number of wavelengths of the surface wave ("phase matching").

For the internal surface waves which have small imaginary parts, one has at resonance ($\omega \cong \mathrm{Re}\,\omega_{nj}$) approximately:

$$(2.11a) \qquad c_j^{ph}(\mathrm{Re}\,\omega_{nj}) \cong a\,\mathrm{Re}\,\omega_{nj}/n,$$

$$(2.11b) \qquad c_j^{gp}(\mathrm{Re}\,\omega_{nj}) \cong a\, d\mathrm{Re}\,\omega_{nj}/dn$$

(to be understood as a difference quotient), so that the dispersion curves of surface waves at the resonance points can be directly read off from the latter.

A similar theory was developed for the case of scattering from spherical [14] and cylindrical [15] inclusions.

3. Numerical results

The (internal) frequency resonances appear in a plot, e. g. of $|S_n^{pp} - 1|$, vs. $k_p a$, as shown in Figure 3.1 (a) for a water-filled cylindrical cavity in aluminium. Here, the resonance and background terms of equation (2.8) interfere; but if the soft-cylinder background is coherently subtracted from S_n^{pp}, one obtains the pure resonances of Figure 3.1 (b). One notices that peaks of the same order j recur in successive modal amplitudes at successively higher frequen-

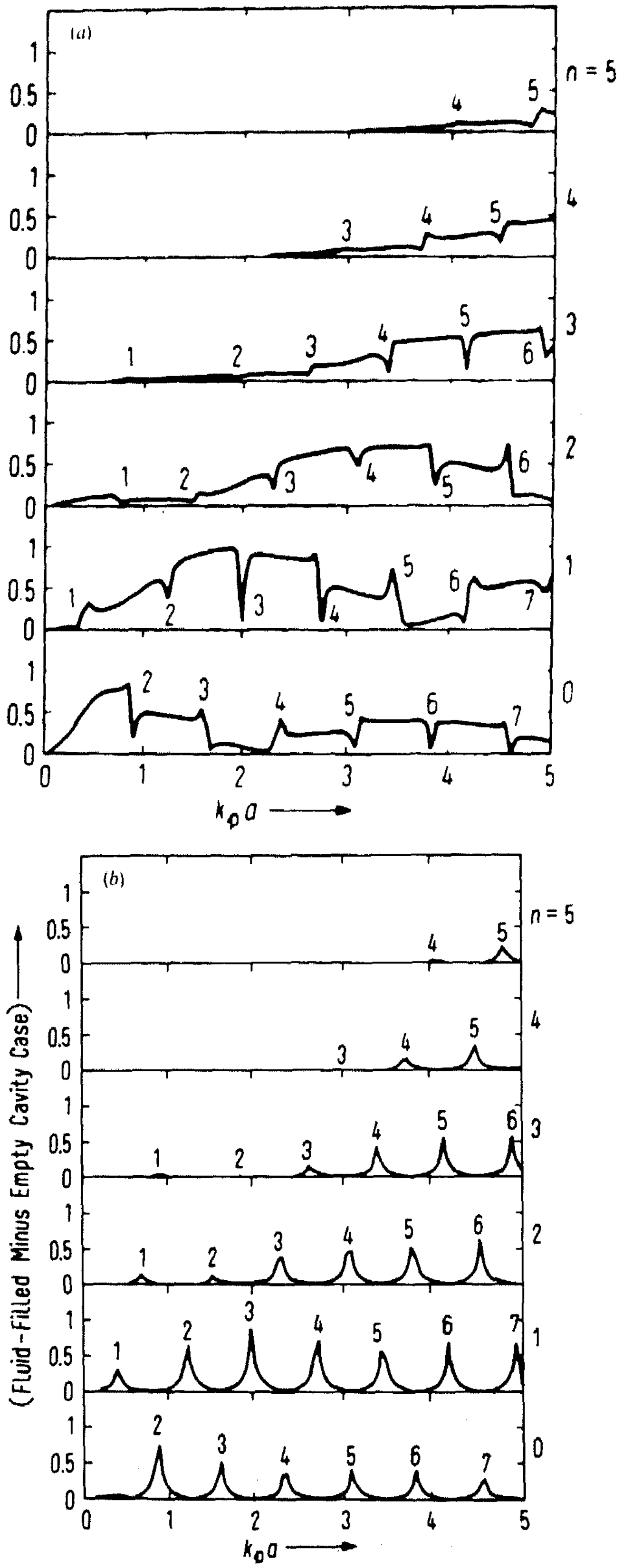

Fig. 3.1. — Modal resonances for a water-filled cylindrical cavity in aluminum: (a) $|S_n^{pp}-1|$ plotted vs. $k_p a$, (b) $|S_n^{pp}-S_n^{(0)\,pp}-1|$, background subtracted.

cies; they represent "Regge recurrences" of the j-th surface wave spanning the cavity's circumference successively with $n=1, 2, 3$ ... wavelengths. The corresponding phase velocity dispersion curves, equation (2.10 a), are shown in Figure 3.2.

The background terms $S_n^{(0)\,ik}$ for an empty cavity themselves have resonances, but with large imaginary parts so that e. g. curves $|S_n^{(0)\,pp}-1|$ appear smooth, as the broad lobes underlying the curves of

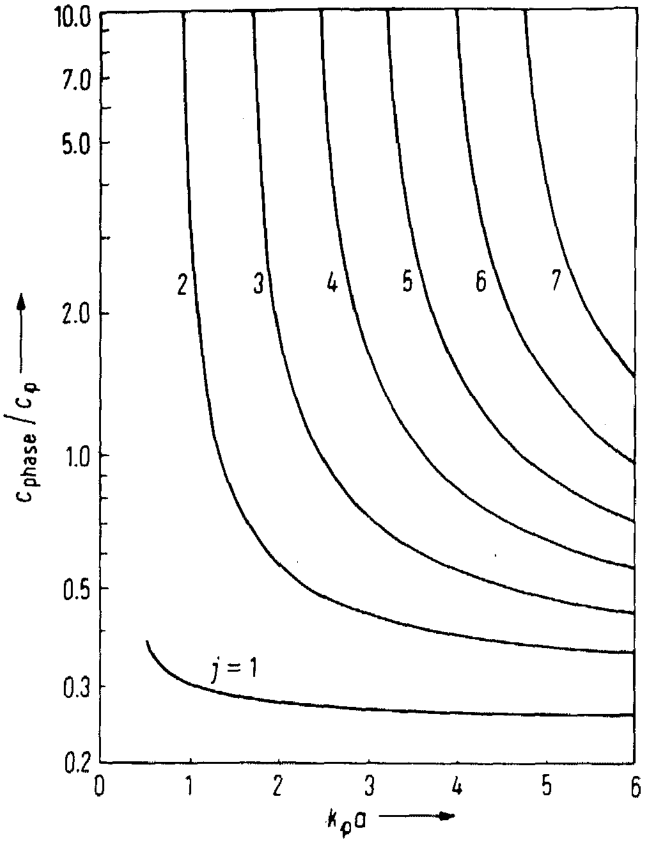

Fig. 3.2. — Dispersion curves of internal surface waves in a water-filled cylindrical cavity in aluminum.

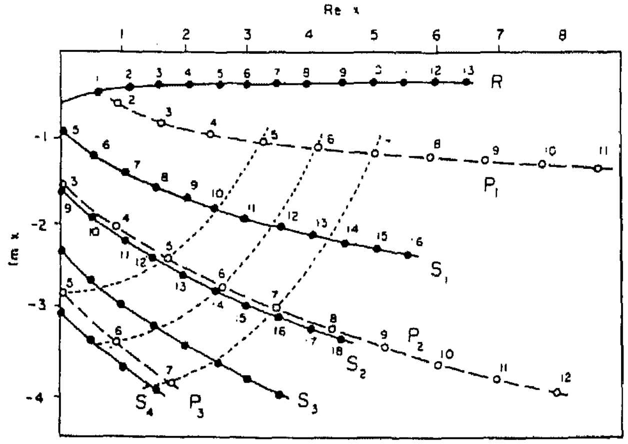

Fig. 3.3. — Complex-frequency poles of scattering amplitudes from a spherical void in steel, showing resonances due to Rayleigh (R), compressional (P) and shear (S) type surface waves.

Figure 3.1(a). One may find their frequency resonances, however, by a complex pole search, as shown in Figure 3.3 for a spherical void in steel [13]. The poles in the $\alpha \equiv k_p a$ plane clearly fall into layers due to Rayleigh (R), compressionnal (P), and shear (S) type (external) surface waves; their dispersion curves are shown in Figure 3.4. Comparing this with Figure 3.2, one sees that dispersion curves for external surface waves start from the origin and reach a plateau for high frequencies, while for internal surface waves, they start at infinity from a low-frequency cutoff, whence they descend to a plateau. This can be understood geometrically.

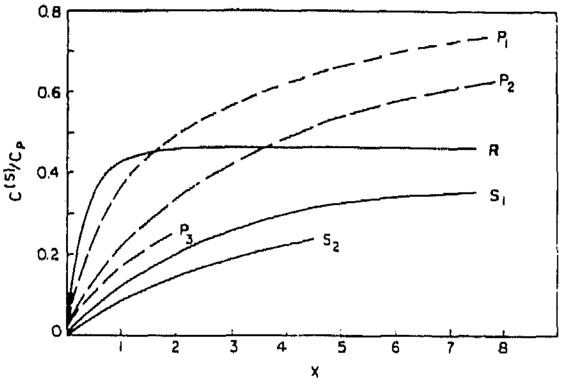

Fig. 3.4. — Dispersion curves of surface waves whose resonances furnish the poles in Figure 3.3.

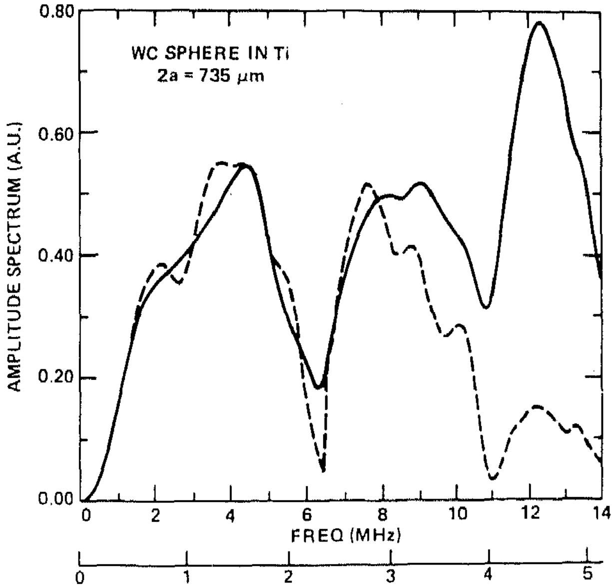

Fig. 3.5. — Theoretical (solid) and experimental (broken curve) backscattering from a spherical tungsten carbide inclusion in titanium.

For a solid inclusion in a solid, results are shown in Figures 3.5 and 3.6. If the total scattering amplitude, e. g., the modulus of equation (2.3), is plotted vs. frequency, a rather smooth curve results (solid curve in *Fig.* 3.5 [22]). If, however, one plots individual modes n in the fashion of Figure 3.1(b), peaks appear (depending, of course, on the materials), which may be used to obtain dispersion curves, Figure 3.6. These indicate the existence of two types of surface waves (solid and dashed), probably also a Rayleigh wave (lowest dashed curve).

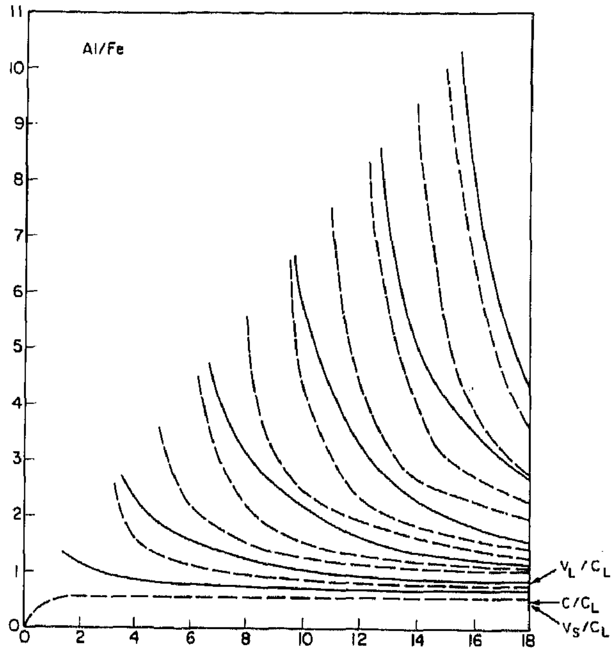

Fig. 3.6. — Dispersion curves for two families of surface waves on an iron sphere in aluminum.

4. Experiments

Experimental results on ultrasonic wave scattering from cavities and inclusions were obtained both in the frequency domain and in the time domain (short pulses) [20-23]. Pulse experiments obtained geometrical and surface wave paths from arrival times. Besides spherical and cylindrical shapes, spheroids have also been investigated. The broken curve in Figure 3.5 shows backscattering results for a spherical tungsten carbide inclusion in titanium [22], as compared to theory.

5. Discussion

The resonance scattering theory (RST) as developed for acoustic scattering from solids, has here been applied to ultrasonic scattering from cavities and inclusions. Internal resonances appear prominently if the properties (e. g. density) of cavity material and its filler differ substantially; if this is not the case, they are less evident. External resonances on cavities are always hard to see; but in all cases (internal and external), resonance frequencies in the complex plane can be calculated no matter how large their imaginary part, and their layers define clearly the various types of surface waves. Experiments have not yet been designed specifically to detect resonances; in a sense, the pulse arrivals indicate these indirectly, being a coherent sum of a number of resonances [17].

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