Helical surface waves on cylinders and cylindrical cavities

Ondes de surface hélicoïdales sur des cylindres et des cavités cylindriques

Herbert ÜBERALL

Department of Physics, Catholic University of America, Washington DC 20064, USA

Professor of Physics, Catholic University of America, Washington DC 20064, USA (since 1964). Field of research: Theoretical Physics (acoustics, ultrasonics, elastic and electromagnetic waves, nuclear physics). *Acoustics:* Advancement of the theory of surface waves in acoustic scattering. Pioneering development of acoustic resonance scattering theory. Channeling in under-ocean sound propagation.

Elastic waves: First development of theory of resonance scattering by cavities and inclusions in solids.

Electromagnetic waves: Establishment of the relation between resonance scattering theory and the Singularity Expansion Method (SEM). Theory of radio-wave propagation in the earth-ionosphere waveguide.

Nuclear theory: Basic development of coherent bremsstrahlung theory (editor of recent book). Development of a model of collective nuclear multipole vibrations, with application to electron scattering (author of two books). Photo-pion reactions. Collective model of ion-ion scattering. Neutrino reactions (editor of book).

SUMMARY

The eigenfrequencies of a cylindrial obstacle (of finite or infinite length) can be interpreted as the resonances due to phase matching of circumnavigating helical surface waves. For the case of a cylinder of finite length, the pitch angle of the helix can assume a discrete set of values only. Resonant eigenvibrations can be excited by waves incident in an oblique fashion, which generates the helical waves. A refraction effect is found to take place between the incident and the helical-wave directions. We obtain pole diagrams of the scattering amplitude in the complex-frequency plane, by using the T-matrix approximation for finite cylinders. In addition, pole diagrams for spheroidal scatterers are obtained by the use of the T-matrix and of spheroidal wave functions. While the poles of symmetric scatterers (spheres or infinite cylinders) are degenerate in the azimuthal quantum number m, the degeneracy for the poles of finite cylinders and of spheroids is lifted. This m-splitting is explained by the phase matching of helical waves with various allowed pitch angles. Dispersion curves for the phase and group velocities and attenuations of the helical waves are obtained

KEY WORDS

Infinite cylinders, finite cylinders, spheroids, eigenfrequencies, resonances, helical surface waves, oblique incidence, complex-frequency poles, m-splitting, dispersion curves, phase velocities, group velocities.

RÉSUMÉ

Les fréquences propres d'un obstacle cylindrique (de longueur finie ou infinie) peuvent être interprétées comme dues à l'accord entre les phases d'ondes se propageant sur la surface d'une façon hélicoïdale. Dans le cas d'un cylindre de longueur finie, l'angle de pas de l'hélice ne peut prendre qu'une série de valeurs discrètes. Des vibrations propres résonnantes peuvent être excitées par des ondes incidentes de direction oblique, ce qui produit les ondes hélicoïdales. Un effet de réfraction est trouvé entre les directions de l'onde incidente et de l'onde hélicoïdale. On obtient des diagrammes de pôles de l'amplitude de diffusion dans le plan complexe de la fréquence, par un calcul utilisant l'approximation de la matrice T pour des cylindres finis. En plus, on obtient des diagrammes de pôles pour des obstacles sphéroïdaux en utilisant la matrice T, ou des fonctions d'ondes sphéroïdales. Tandis que les pôles d'obstacles symétriques (sphères, ou cylindres infinis) dégénérent vis-à-vis du nombre quantique azimuthal m, cela n'est plus le cas pour les pôles de cylindres finis et de sphéroïdes. La séparation résultante entre les valeurs de m s'explique alors par l'accord de phases des ondes hélicoïdales possédant différents angles d'inclinaison permis. On obtient des courbes de dispersion pour les vitesses de phase et de groupe des ondes hélicoïdales.

MOTS CLÉS

Cylindres finis, cylindres infinis, sphéroïdes, fréquences propres, résonances, ondes de surface hélicoïdales, incidence oblique, pôles de fréquence complexe, écartement (dédoublement) en m, courbes de dispersion, vitesse de phase, vitesse de groupe.

CONTENTS

1. Introduction

The most general case of the scattering of waves obliquely incident on an infinite cylinder has first been considered by White [1]; he deals with a plane wave in an elastic medium incident on an elastic cylindrical inclusion. This theory has recently been completed by Delsanto *et al.* [2, 3]; it is based on the classical normal-mode (or partial-wave) Rayleigh series expansion. Special cases have been investigated in greater detail: acoustic waves incident on an elastic cylinder [4] or shell [5], or on a fluid cylinder [6, 7], and elastic waves incident on a fluid-filled cylindrical cavity [8] (in a similar fashion, elastic waves on a spherical cavity have also been studied [9]). In these latter investigations, the complex eigenfrequencies of the mentioned obstacles have been explicitly calculated, as well as the excitation of the corresponding eigenvibrations. While analogous calculations cannot be carried out for the case of cylinders of finite length, due to the non-separability of this problem, results for the eigenfrequencies can nevertheless be obtained here by the use of special methods. If only the interior vibrations of a fluid cylinder in vacuo are considered, the problem is still exactly soluble, and the corresponding eigenvibrations have, for the first time, been interpreted as being due to the phase matching of helical surface waves [10] (of internal type, in this case). For the non-separable exterior problem of finite-length impenetrable cylinders, the T-matrix method of Waterman [11] has been employed, modified so as to furnish complex eigenfrequencies [12]. A very comprehensive study of this probelm has recently been carried out, on the basis of this method, which is contained in a paper [13] that also discusses the acoustic eigenfrequencies of impenetrable spheroids obtained by the use of spheroidal functions. In this study, the phase matching of external helical surface waves has been invoked as an explanation for the finite-cylinder eigenfrequencies, and for their splitting into components corresponding to different values of the azimuthal quantum number *m* (the latter being a measure for the pitch angles of the helical waves, which due to the finite cylinder length form a discrete set). As to the excitation of helical surface waves by incident sound, if was found [6,13] that this takes place in a refractive way, the helical pitch angle being different from the indident angle. Experiments on the scattering of obliquely incident sound by elastic cylinders have recently been initiated [14].

2. Internal helical waves

A fluid cylinder in vacuo, of radius a and length L admits an internal acoustic field (wave vector **k**, with $k=\omega/c$):

(2.1) $$p(\mathbf{r})=J_m(Kr)\,e^{\pm ik_\varphi a\varphi}\,e^{\pm ik_z z},$$

where:

(2.2 a) $$K^2=k^2-k_z^2;$$

the single-valuedness condition under $\varphi \rightarrow \varphi + 2\pi$ gives:

(2.2 *b*) $\qquad k_\varphi = m/a, \qquad m = 1, 2, \ldots,$

and the boundary condition at the end faces leads to:

(2.2 *c*) $\qquad k_z = j\pi/L, \qquad j = 1, 2, \ldots.$

The Dirichlet boundary condition at $r = a$ gives:

(2.2 *d*) $\qquad K = x_{mn}/a, \qquad n = 1, 2, \ldots,$

where x_{mn} is the n-th zero of $J_m(x)$. [The Neumann condition for a fluid cylinder in a rigid enclosure would lead to the zero of $J'_m(x)$.] Inserted in equation (2.2 *a*), this gives the eigenvalues k_{nmj} of k.
We now introduce a tangential wave vector $\mathbf{k}_s$ of the surface field, with:

(2.3) $$k_s^2 = k_\varphi^2 + k_z^2,$$

which describes the propagation of helical surface waves. The conditions (2.2 *b*) and (2.2 *c*) then represent the phase matching of such waves after circumnavigating the cylinder, and/or getting reflected from the end faces. Equations (2.2 *b*), (2.2 *c*) inserted in equation (2.3) furnish eigenvalues $(k_s)_{mj}$.
The helical-wave phase velocities $c_s = \omega/k_s$ can be obtained from $c_s/c = k/k_s$ at the discrete points (resonance frequencies) where phase matching is satisfied:

(2.4) $$(c_s/c)_{nmj} = \{[x_{mn}^2 + (j\pi)^2 (a/L)^2] / [m^2 + (j\pi)^2 (a/L)^2]\}^{1/2}.$$

Note that these correspond to helical waves of pitch angle α with the z-axis, $\tan\alpha = k_\varphi/k_z$, which due to the finite length of the culinder assumes the discrete values:

(2.5) $$\tan\alpha_{mj} = (m/j\pi)(L/a).$$

A given helical wave thus corresponds to a fixed ratio m/j. For an infinite cylinder ($L \rightarrow \infty$, $j \rightarrow \infty$) α is continuous.

Figure 2.1 shows how the discrete points of equation (2.4), corresponding to the eigenfrequencies of the cylinder, when connected according to equation (2.5) furnish the dispersion curves of the helical surface waves.

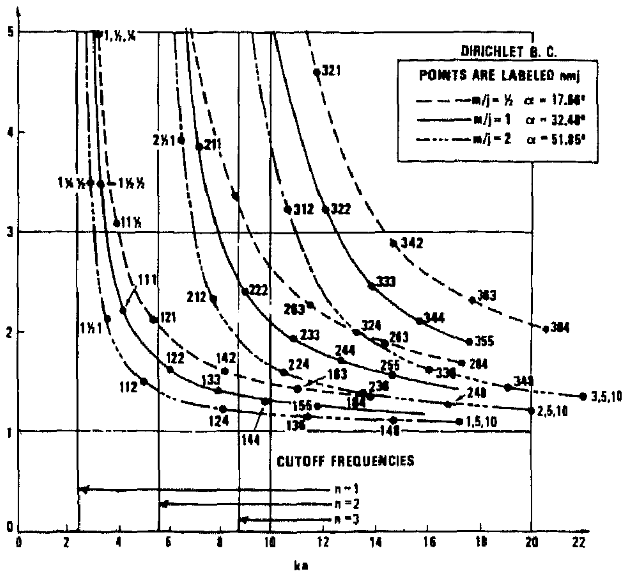

Fig. 2.1. – Dispersion curves of internal helical surface waves on a finite fluid cylinder in vacuo, of dimension 1:1 ($L = 2a$).

3. Refraction effect

For a cylinder in a medium helical waves can be generated by an incident plane acoustic wave. If the latter arrives at an angle γ with the z-axis so that $k_y = k \sin\gamma$, $k_z = k\cos\gamma$, the total field for an infinite cylinder, given by equation (2.2 *c*) of reference [15], gets modified to:

(3.1) $$p = \frac{1}{2} e^{ik_z z} \sum_{n=0}^{\infty} (2 - \delta_{n0})\, i^n \{H_n^{(2)}(k_y r) + S_n H_n^{(1)}(k_y r)\} \cos n\varphi,$$

leading after application of the Watson transformation to the creeping-wave sum analogous to equation (2.10 *c*) of [15]:

(3.2) $$p_{cw} = \pi e^{ik_z z} \sum_{l=1}^{\infty} \frac{e^{i\pi\nu_l/2}}{\sin\pi\nu_l} \times S_{\nu_l}^{(R)} H_{\nu_l}^{(1)}(k_y r) \cos\nu_l(\varphi - \pi).$$

The phase factor $\Phi_l = k_z z + \varphi \operatorname{Re}\nu_l$ shows that these surface waves are helical, with wave fronts $a\varphi = -(ak_z/\operatorname{Re}\nu_l)\, z$ whose normals make an angle $\varphi_l = \tan^{-1}(\operatorname{Re}\nu_l/ak_z)$ with the z-axis. This defines the law of refraction:

(3.3) $$\tan\varphi_l = g_l \tan\gamma, \qquad g_l = \operatorname{Re}\nu_l/k_y a,$$

between incident direction γ and helical-wave direction φ_l. The phase velocity $v_l^{ph} = ck/k_l$ of the helical waves is:

(3.4) $$v_l^{ph} = c/\{(\operatorname{Re}\nu_l/ka)^2 + \cos^2\gamma\}^{1/2};$$

for the case of external waves on rigid or soft cylinders where the asymptotic expansion of Franz [16] for $\nu_l(k_y a)$ can be used, one has:

(3.5) $$v_l^{ph} \cong c\Big/\left\{1 + \frac{q_l}{2.6^{1/3}}\left(\frac{\sin^2\gamma}{ka}\right)^{2/3} + \ldots\right\},$$

where $q_1^r = 1.469\,354$, $q_1^s = 3.372\,134$. Including higher terms, figure 3.1 shows v_l^{ph} of helical waves at $\gamma = 0°$ and 45°, and the refraction angle φ_l for $\gamma = 45°$, for a soft cylinder.

4. Complex eigenfrequencies

The eigenfrequencies k_{nmj} for a cylinder in vacuo, Section 3, are real since no radiation loss can occur. We present figures showing examples of complex

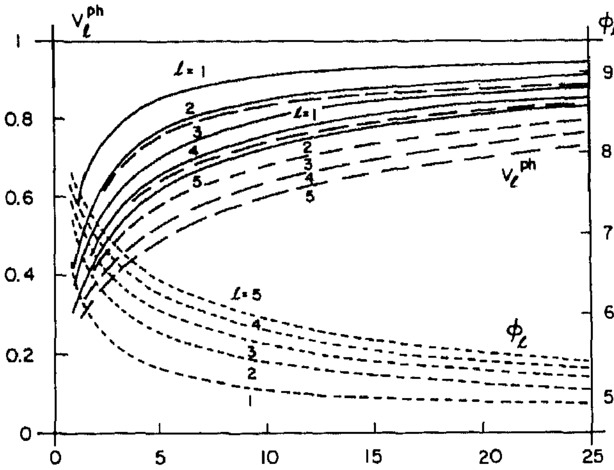

Fig. 3.1. — Dispersion curves of external helical surface waves on an infinite soft cylinder for $\gamma=90°$ (dashed) and 45° (solid), and refraction angle φ_l at $\gamma=45°$ (right scale), plotted vs. $k_y a$.

eigenfrequencies that correspond to the resonances, due to phase matching, of external circumferential waves on elongated objects.

Figure 4.1 shows the eigenfrequencies in the complex kb plane of an acoustically rigid prolate spheroid, for various axis ratios b/a as indicated (a being the semiminor axis). These were obtained [13] by satisfying the boundary condition with spheroidal wave functions. One notices that the sphere values

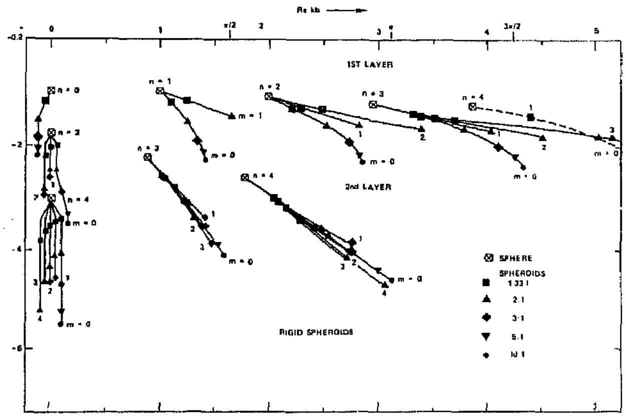

Fig. 4.1. — Complex eigenfrequencies for a rigid spheroid with semi-major axis b.

(crossed squares), degenerate in m, split into branches non-degenerate in m for spheroids. Figure 4.2 shows the same for the magnetic eigenvibrations of a conducting cylinder of length L (the electromagnetic analogue of an acoustically rigid cylinder) obtained with the T-matrix method [11-13]. A phase-matching model for surface waves has been developed here [13] in order to show that the m-splitting of the eigenfrequencies corresponds to helices of different pitch angles.

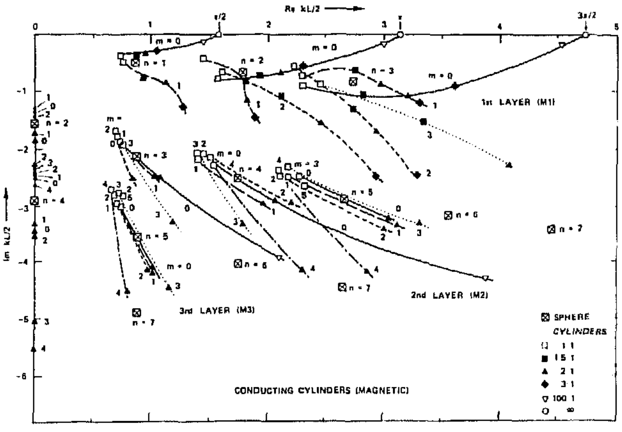

Fig. 4.2. — Complex magnetic eigenfrequencies of a conducting cylinder of length L.

5. Cavities

Analogous results were obtained for infinite cylindrical cavities, using our general theory [2, 3]. Figure 5.1 shows the eigenfrequencies in the complex $k_y a$ plane for an empty cavity in aluminum, for both compressional (p) and shear (s) type surface waves (here, k is the dilatational propagation constant).

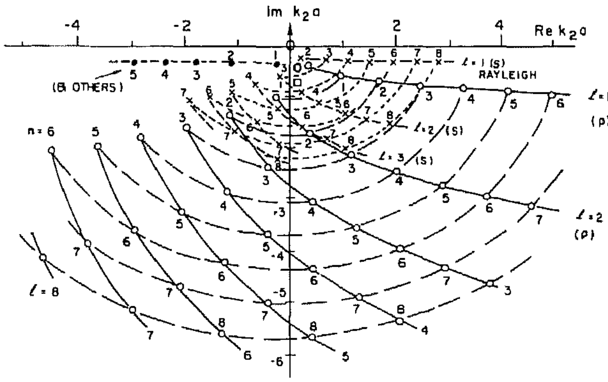

Fig. 5.1. — Eigenfrequencies in the complex $k_y a$ plane for an empty cylindrical cavity in aluminum.

6. Conclusion

The complex eigenfrequencies of finite-length cylinders show a splitting according to the azimuthal quantum number m. They can be interpreted as the resonances, due to phase matching, of helical surface waves of different pitch angles. For infinite cylinders, a continuum of pitch angles occurs. The helical surface waves can be excited by incident acoustic waves, and refraction takes place between the incident and the surface wave directions. Recent experiments [14] are now investigating these problems. Theoretically, there have also been geometrical investigations of helical waves on cylinders [17].

Portions of this work were supported by the Office of Naval Research, the Army Research Office, the Naval Surface Weapons Center, and the David Taylor Ship Research and Development Center.

REFERENCES

[1] R. M. White, Elastic wave scattering at a cylindrical discontinuity in a solid, *J. Acoust. Soc. Amer.*, 30, 1958, p. 771-785.

[2] P. P. Delsanto, J. D. Alemar, E. Rosario, J. V. Subrahmanyam, A. Nagl, H. Überall and J. Valcárcel, Resonances and surface waves in elastic wave scattering from cavities and inclusions, *Review of Progress in Quantitative Non-destructive Evaluation*, D. O. Thompson and D. E. Chimenti, Eds., Plenum Press., New York, 3 A, 1984, p. 111-121.

[3] P. P. Delsanto, J. D. Alemar, E. Rosario, A. Nagl, J. V. Subrahmanyam and H. Überall, Elastic wave scattering from cylindrical cavities and solid inclusions analyzed by the resonance method, *Transaction of the First Army Conference on Applied Mathematics and Computing*, George Washington University, Washington DC, May 1983 (ARO Report 84-1), p. 773-784.

[4] L. Flax, V. K. Varadan and V. V. Varadan, Scattering of an obliquely incident acoustic wave by an infinite cylinder, *J. Acoust. Soc. Amer.*, 68, 1980, p. 1832-1835.

[5] J. V. Subrahmanyam, Creeping wave analysis through frequency plane for an obliquely incident plane wave on an elastic scatterer, *Ph. D. thesis*, Catholic University of America, Washington DC, 1983.

[6] A. Nagl, H. Überall, P. P. Delsanto, J. D. Alemar and E. Rosario, Refraction effects in the generation of helical surface waves on a cylindrical obstacle, *Wave Motion*, 5, 1983, p. 235-247.

[7] J. D. Alemar, P. P. Delsanto, E. Rosario, A. Nagl and H. Überall, Spectral analysis of the scattering of acoustic waves from a fluid cylinder I, II and III, *Acustica*, submitted.

[8] P. P. Delsanto, J. D. Alemar, E. Rosario, A. Nagl and H. Überall, Spectral analysis of the scattering of elastic waves from a fluid-filled cylinder, in *Transactions of the second Army Conference on Applied Mathematics and Computing*, Rensselaer Polytechnic Institute, Troy NY, May 1984.

[9] A. Nagl, Y. J. Stoyanov, J. V. Subrahmanyam, H. Überall, P. P. Delsanto, J. D. Alemar and E. Rosario, Surface wave modes on spherical cavities excited by incident ultrasound, in *Review of Progress in Quantitative Nondestructive Evaluation*, 4 A, D.O. Thompson and D.E. Chimenti, Eds., Plenum Press, New York, 1985, p. 161-165.

[10] G. C. Gaunaurd, E. Tanglis and H. Überall, Surface wave interpretation of the eigenfrequencies of a finite-length fluid cylinder, *J. Acoust. Soc. Amer.*, 67, 1980, p. 764-769.

[11] P. C. Waterman, Numerical solutions of electromagnetic scattering problems, in *Computer Techniques for Electromagnetics.*, R. Mittra, Ed., Pergamon Press, New York, 1973, 7, Chapt. 3.

[12] P. J. Moser, The isolation, identification, and interpretation of resonances in the radar scattering cross section for conducting bodies of finite general shape, *Ph. D. thesis*, Catholic University of Americal, Washington DC, 1982.

[13] H. Überall, P. J. Moser, Barbara L. Merchant, A. Nagl, K. B. Yoo, S. H. Brown, J. W. Dickey and J. M. D'Archangelo, Complex acoustic and electromagnetic resonance frequencies of prolate spheroids and related elongated objects, and their physical interpretation, *J. Appl. Phys.*, 58, 1985, p. 2109-2124 P. J. Moser, H. Überall, J. M. D'Archangelo, Barbara L. Merchant, A. Nagl and K. B. Yoo, Complex eigenfrequencies of axisymmetric objects: physical interpretation in terms of resonances, *Proceed. IEEE*, 72, 1984, p. 1652-1653.

[14] G. Maze, J. L. Izbicki and J. Ripoche, Diffusion ultrasonore par un cylindre insoné obliquement, *Revue du CETHEDEC*, 78, 1984, p. 95-106.

[15] H. Überall, Acoustic scattering from elastic cylinders and spheres: surface waves (Watson transform) and transmitted waves, this colloquium.

[16] W. Franz, Über die Greenschen Funktionen des Zylinders und der Kugel, *Z. Naturforsch. A*, 9, 1954, p. 705-716.

[17] J. Perdijon, Miroir pour l'examen d'un objet cylindrique sous une incidence constante le long d'hélices données, *Rev. Phys. Appl.*, 10, 1975, p. 319-324.