t-designs with repeated points in Q-polynomial association schemes

t-designs avec points répétés dans des schémas d'association Q-polynomial

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L'auteur a fait des études en mathématiques à EUT, Eindhoven, Pays-Bas, avec spécialisation mathématiques discrètes, combinatoire et codage (équipe de v. Lint) et l'article fait partie de sa thèse « masters degree » (sorte de « thèse de troisième cycle »).
En mai 1982 l'auteur est entré au CNET, Issy-les-Moulineaux, pour faire de la recherche dans le domaine de traitement de signaux, plus précisément sur les transformations en nombres entiers (transformations de Fourier généralisés), en collaboration avec M. Duhamel.
Ses champs d'intérêt sont d'un côté la recherche théorique en mathématiques discrètes, d'un autre côté les applications précises de ces résultats dans la pratique.
Une thèse d'état est en préparation, ainsi qu'une maquette d'un convolueur rapide.

SUMMARY

In this paper, the Rao-Wilson bound [1], together with the dual of Lloyds Theorem are generalised to t-designs with repeated points in Q-polynomial association schemes. The proof uses a generalisation of a result of Connor [5] for classical 2-designs. Moreover, a new proof is given of a sharper version of McWilliams inequality, and the case of equality is treated. With $e=[t/2]$, the generalised Rao-Wilson bound becomes $b \geqq \Phi(y) \sum_{j=0}^{e} u_j$, where b denotes the total number of points and $u_0, \ldots, u_n$ are the multiplicities of the scheme, if some point y is repeated $\Phi(y)$ times. Specializing to Johnson- and Hamming-schemes, we find $b \geqq e_i \binom{v}{e}$ for classical t-designs on v points having b blocks, if some block i is repeated e_i times (*see* [4]), and $b \geqq e_i \sum_{j=0}^{e} \binom{n}{j}(q-1)^j$ for orthogonal arrays of maximum strength t and length n, over a q-letter alphabet, having b rows, if some row i is repeated e_i times.

KEY WORDS

Association scheme, t-designs with repeated blocks, orthogonal arrays with repeated lines, Rao-Wilson bound.

RÉSUMÉ

Dans cet article, la borne de Rao-Wilson [1], *ainsi que le dual du théorème de Lloyd, sont généralisés aux t-designs à points répétés dans les schémas d'association Q-polynomiaux. La démonstration utilise une généralisation d'un résultat de Connor* [5] *pour les 2-designs classiques. De plus, on donne une nouvelle démonstration de l'inégalité de McWilliams dans une version légèrement plus forte, et on traite le cas de l'égalité. Avec* $e=[t/2]$, *la borne généralisée de Rao-Wilson devient* $b \geqq \Phi(y) \sum_{j=0}^{e} u_j$, *où b est le nombre total de points, et* $u_0, u_1, \ldots, u_n$ *sont les multiplicités du schéma, si un point y est répété* $\Phi(y)$ *fois. Se restreignant aux schémas de Johnson et de Hamming, on trouve* $b \geqq e_i \binom{v}{e}$, *pour des t-designs classiques à b blocs, sur v points, si un bloc i est répété* e_i *fois* ([4]), *et* $b \geqq e_i \sum_{j=0}^{e} \binom{n}{j} (q-1)^j$, *pour des tableaux orthogonaux de force maximale t, à b lignes, de longueur n, sur un alphabet à q lettres, si une ligne i est répétée* e_i *fois.*

MOTS CLÉS

Schéma d'association, t-designs aux blocs répétés, tableaux orthogonaux aux lignes répétées, borne de Rao-Wilson.

CONTENTS

Introduction

The theory of P- and Q-polynomial association schemes is of interest both to coding theory and design theory. Indeed, both the Hamming scheme (which provides a setting for coding theory) and the Johnson scheme (which provides a setting for design theory) are P- and Q-polynomial. Delsarte [1] was the first to make a systematic use of these facts to prove in a unified way a number of hitherto unconnected results from design and coding theory.

He showed in particular that his concept of a t-design in Q-polynomial schemes generalises the classical t-designs (in Johnson schemes) and the orthogonal arrays of maximum strength t (in Hamming schemes), and proved the Rao-Wilson bound $|Y| \geqq u_0 + u_1 + \ldots + u_e$ for a t-design Y (where $e=[t/2]$ and $u_0, u_1, \ldots, u_n$ are the multiplicities of the scheme) together with a « dual » of Lloyd's Theorem in case of equality.

The Rao-Wilson bound reduces to $b \geqq \binom{v}{e}$ (Wilson-Petrenjuk) for classical t-designs with b blocks on v points, and to $b \geqq \sum_{i=0}^{e} \binom{n}{i} (q-1)^i$ (Rao) for orthogonal arrays with b rows, of wordlength n over a q-letter alphabet. In [1] Delsarte also introduced t-designs with repeated points in Q-polynomial schemes, and this generalises classical t-designs with repeated blocks and orthogonal arrays with repeated rows.

There has been some interest in classical t-designs with repeated blocks. In particular, Mann [2] generalised Fisher's inequality for 2-designs to $b \geqq e_i v$ if some block is repeated e_i times (*see* also [3]) and this was generalised further by Wilson and Ray-Chaudhury to $b \geqq e_i \binom{v}{e}$ for t-designs.

This paper is constructed as follows: after section 1, which introduces association schemes and some notation, section 2 contains a more or less straightforward generalisation of [1], section 3.5. In section 3 we give a new proof of a sharper form of McWilliams inequality (*see* [1], section 5.1) and discuss the case of equality. Then, in section 4, we generalise the Rao-Wilson bound to $b \geqq \Phi(y)(u_0 + u_1 + \ldots + u_e)$, where b denotes the "total number of points" in the t-design and $\Phi(y)$ denotes the number of occurrencies of the point y. Moreover, a result of Connor for 2-designs (*see* [3], [5]) is generalised and this is used to obtain a "dual" of Lloyds Theorem in case of equality.

Most of the results in this paper are part of the authors masters thesis [6], where also missing details can be found.

1. Association schemes: Definitions and notations

We recall the following from [1]:

Let X be a finite set and $\Gamma = \{\Gamma_0, \ldots, \Gamma_n\}$ a set of $n+1$ relations on X.

(1.1) **Definition:** The pair (X, Γ) is called (*symmetric*) *association scheme* with n classes iff:

(i) Γ is a *partition* of $X \times X$ and Γ_0 is the diagonal relation, i.e. $\Gamma_0 = \{(x, x) \mid x \in X\}$;

(ii) each relation Γ_k is *symmetric*, i.e. $(x, y) \in \Gamma_k$ iff $(y, x) \in \Gamma_k$;

(iii) for all $(x, y) \in \Gamma_k$, the number :

$$p_{ij}^k = |\{z \in X \mid (x, z) \in \Gamma_i, (z, y) \in \Gamma_j\}|,$$

depends only on i, j and k ($i, j, k = 0, 1, \ldots, n$).

So an n-class symmetric association scheme can be seen as a colouring of the complete graph K_X with n colours, such that the number of triangles with a given colouring on a given base depends only on the colouring and not on the base.

Let D_i denote the *adjacency matrix* of Γ_i, i.e. $D_i(x, y)$ equals 1 or 0 according to whether (x, y) is in Γ_k or not.

It can be shown that the matrices $D_0, \ldots, D_n$ span a commutative $(n+1)$ dimensional subalgebra of $\mathbb{R}(X, X)$, called the *Bose-Mesner* (*BM*) *algebra* of the scheme. Moreover, such an algebra admits a base of $(n+1)$ mutually orthogonal symmetric *idempotents*, denoted by $J_0, \ldots, J_n$. Also, the BM-algebra can be simultaneoustly diagonalised by a matrix S in $\mathbb{R}(X, X)$ and this matrix S can be partitioned as $S = [S_0 S_1 \ldots S_n]$ such that $J_k = |X|^{-1} S_k S_k^T$.

The first and second *eigenmatrices* of the scheme shall be denoted by P and Q. P and Q are both real $(n+1) \times (n+1)$ matrices. (In fact, the k-th column of P consists of the eigenvalues of D_k and Q is then defined by $PQ = QP = |X| I$.)

Their importance in the theory of association schemes stems from the fact that they can be computed from the parameters p_{ij}^k of the scheme.

We shall denote the k-th column of Q(P) by $Q_k(P_k)$ $(k = 0, \ldots, n)$. The numbers $v_k := P_k(0)$ and $u_k := Q_k(0)$ are called the *valencies* and *multiplicities* of the scheme.

Let there be given $n+1$ distinct non-negative numbers $z_0, z_1, \ldots, z_n$. Then there exists $n+1$ polynomials $\varphi_0(z), \ldots, \varphi_n(z)$ in $\mathbb{R}[z]$ (i.e. with real coefficients) such that $\varphi_k(z_i) = Q_k(i)$ $(k, i = 0, 1, \ldots, n)$.

(1.2) **Definition:** If for all k, the polynomial $\varphi_z(z)$ defined above has degree k, then we call (X, Γ) *Q-polynomial* wih respect to $z_0, \ldots, z_n$.

P-*polynomial* schemes are defined analogously. It can be shown that a scheme is P-polynomial iff it is *metric*, i.e. iff the function d on $X \times X$ defined by $d(x, y) = i$ iff $(x, y) \in \Gamma_i$ is a metric on X. (This rather surprising fact is one of the many examples in the theory of an interconnection between algebraical and combinatorial properties.)

(1.3) **Definition:** A vector $\Phi \in \mathbb{R}(X)$ with $\Phi(x) \geqq 0$ for all $x \in X$ is called a *design*.

The set $Y := \{x \in X \mid \Phi(x) > 0\}$ is the *support* of Φ, $b := \sum_{x \in X} \Phi(x)$ is the *total number of points* of Φ. The *inner-distribution* of Φ is the vector $a = (a_0, \ldots, a_n)^T$ defined by $a_k := b^{-1} \Phi^T D_k \Phi$ $(k = 0, \ldots, n)$ and the *outer-distribution* of Φ is the matrix $B = [D_0 \Phi, \ldots, D_n \Phi]$.

Remark that a design Φ can be seen as a subset Y of X with multiplicities $\Phi(y)$ accorded to each point y in Y. Then $B(x, k)$ is the number of points in Y (counted according to their multiplicities) in relation k with x in X an a_k is the average of $B(y, k)$ over y in Y.

For any vector $w = (w_0, \ldots, w)^T$ in $\mathbb{R}^{n+1}$, we define: $s(w)$ is the number of $i \neq 0$ such that $w_i \neq 0$, and $t(w)$ is the largest t such that $w_1 = w_2 = \ldots = w_t = 0$.

(1.4) **Definition:** If (X, Γ) is Q-polynomial, and Φ a design with inner-distribution a, then the *degree s* and the *maximum strength t* of Φ are $s := s(a)$, $t := t(a^T Q)$.

We shall say that Φ is a *t-design of degree s* in (X, Γ).

One of the main aims of the theory is to obtain bounds on subsets (or on the total number of points in designs) whose inner distribution satisfy certain properties. To this end, the following Theorem is fundamental ([1], (3.8)):

(1.5) (*Linear programming bound*):

$$BQ_k = |X| J_k \Phi,$$
$$a^T Q = |X| b^{-1} \|J_k \Phi\|^2 \geqq 0.$$

Some examples of association schemes are:

– the *Hamming scheme* $H(n, q)$. Here the set X consists of all words of length n over a q-letter alphabet and two words are in relation Γ_i iff they have Hamming-distance i. The multiplicities are given by $u_i = \binom{n}{i}(q-1)^i$;

– the *Johnson scheme* $J(n, v)$. Here the set X consists of all n-subsets of a fixed v-set, and for $A, B \in X$, $(A, B) \in \Gamma_i$ iff $|A \cap B| = n - i$.

The multiplicities are given by $u_i = \binom{v}{i} - \binom{v}{i-1}$.

Both the Hamming- and Johnson schemes are P- and Q-polynomial. Moreover, the concept of a t-design Φ in Q-polynomial schemes as defined above, coincides for Φ a 0-1 vector [or $\Phi \in \mathbb{Z}(X)$] with that of *classical t-designs* in Johnson schemes (with repeated blocks) and with that of *orthogonal arrays of maximum strength t* in Hamming schemes (with repeated rows).

The reader not familiar with association schemes is adviced to skip section 2 (at least on first reading).

2. Idempotents from t-designs

For the rest of this paper, let (X, Γ) be an n-class association scheme, Q-polynomial with respect to the points $z_0=0, z_1, \ldots, z_n$ and the polynomials $\varphi_0(z)=1, \varphi_1(z), \ldots, \varphi_n(z)$. The *sum-polynomials* (*Wilson-polynomials*) $\psi_k (k=0, \ldots, n)$ are

$$\psi_k(z)=\sum_{i=0}^{k} \varphi_i(z).$$

Moreover, let $\Phi \in \mathbb{R}(X)$ be a design with support $Y \subseteqq X$, of maximum strength t [note that $\Phi(y)>0$ iff $y \in Y$ by definition] with associated inner distribution $a \in \mathbb{R}^{n+1}$.

The total number of points b of Φ is $b=\sum_{x \in X} \varphi(x)$.

Let $e:=[t/2]$.

This section is devoted to the generalisation of [1], section 3.5.

So let $H_k (k=0, 1, \ldots, n)$ be the k-th characteristic matrix of Y (*see* [1]). We define the diagonal matrix $\Delta \in \mathbb{R}(X \times X)$ by $\Delta=\operatorname{diag}(\Phi)$, and $\bar{\Delta}$ is the restriction of Δ to $Y \times Y$.

Moreover, let $L_k:=\bar{\Delta}^{1/2} H_k$.

The following three theorems are given in [1] for the case that Φ is a 0-1 vector (i.e. for a set Y without repeated points).

(2.1) **Theorem:**

(i) $L_i^T L_j = S_i^T \Delta S_j \qquad (i, j=0, 1, \ldots, n)$;

(ii) $\| L_i^T L_j \|^2 = b \sum_{k=0}^{n} q_{ij}^k a^T Q_k \qquad (i, j=0, 1, \ldots, n)$;

(iii) $$L_k L_k^T = \sum_{i=0}^{n} Q_k(i) \bar{\Delta}^{1/2} (D_i | Y) \bar{\Delta}^{1/2} = |X| \bar{\Delta}^{1/2} (J_k | Y) \bar{\Delta}^{1/2} \qquad (k=0, 1, \ldots, n).$$

Proof: As in [1], Thm. (3.13) and (3.14). □

(2.2) **Theorem:**

$$L_i^T L_j = \begin{cases} 0 & \text{if } i \neq j, \ i, j \leqq e, \\ b I & \text{if } i=j \leqq e. \end{cases}$$

Proof: As in [1], Thm. (3.15). Note that $a^T Q_k=0$ for $k=1, \ldots, t$ and $q_{ij}^k=0$ if $i, j \leqq e$, $k>2e$ by the Q-polynomial property. □

Let us now define the matrices $E_0, E_1, \ldots, E_{e+1} \in \mathbb{R}(Y \times Y)$ by:

(2.3) **Definition:**

$$E_i=\frac{1}{b} L_i L_i^T \qquad (i=0, 1, \ldots, e),$$

$$E_{e+1}=I-E_0-\ldots-E_e.$$

Then the main results of this section is:

(2.4) **Theorem:** $E_0, E_1, \ldots, E_{e+1}$ *are mutually orthogonal symmetric idempotents in* $\mathbb{R}(Y \times Y)$, *of rank* $u_0, u_1, \ldots, u_e, |Y|-u_0-u_1-\ldots-u_e$.

Proof: The first part follows directly from (2.2).

To find the ranks, note that:

$$\operatorname{rank}(E_k)=\operatorname{TR}(E_k)=\frac{1}{b}\operatorname{TR}(L_k L_k^T) = \frac{1}{b}\operatorname{TR}(L_k^T L_k)=\operatorname{TR}(I_{u_k})=u_k. \ \square$$

3. The McWilliams inequality

The results of this section belong to the theory of orthogonal polynomials. Their importance for Q-polynomial schemes stems from the connection of the polynomials $\varphi_k(z)$ with the Q-matrix of the scheme.

We recall the following facts from [1], section 5:

Let $\mathbb{R}_n[z]$ denote the set of real polynomials of degree at most n, and let the inner-products (,) and [,] on $\mathbb{R}_n[z]$ be defined as:

$$(f, g)=\sum_{i=0}^{n} v_i f(z_i) g(z_i),$$

$$[f, g]:=\sum_{i=1}^{n} v_i z_i f(z_i) g(z_i).$$

Then the polynomials $\varphi_0, \varphi_1, \ldots, \varphi_n$ are orthogonal with respect to (,) and the sumpolynomials $\psi_0, \ldots, \psi_n$ are orthogonal with respect to [,].

(Remember that $\psi_k(z)=\sum_{k=0}^{k} \varphi_k(z)$).

Let ψ_e have zeroes $p_1, \ldots, p_e$. The k-th Christoffel-number w_k of ψ_e corresponding to p_k is defined by:

$$w_k^{-1}=\sum_{j=0}^{e-1} (\psi_j(p_k)/\sigma_j')^2$$

(where $\sigma_j'^2=[\psi_j, \psi_j]$).

The importance of the Christoffel-numbers stems from Theorem (3.1) below, which is slightly more general than [1], Thm. (5.4).

(3.1) **Theorem:** *Let $m \in \mathbb{N}$, $b_1, \ldots, b_s$ and $q_1, \ldots, q_s$ be given, such that:*

$$[f, 1] = \sum_{j=1}^{s} b_j f(q_j) \quad \textit{for all } f(z) \in \mathbb{R}_m[z].$$

Then $m \leqq 2s-1$ and equality holds iff $q_1, \ldots, q_s$ are the zeroes of ψ_s and $b_1, \ldots, b_s$ are the corresponding Christoffel numbers of ψ_s.

Proof: Define:

$$g(z) := (q_1 - z)(q_2 - z) \ldots (q_s - z).$$

If $m = 2s$ then $[g, g] = [g^2, 1] = 0$, which is impossible since $g \neq 0$. So $m \leqq 2s-1$.

If $m = 2s-1$ then for $k = 0, 1, \ldots, s-1$ we have $[g, \psi_k] = [g\,\psi_k, 1] = 0$. Since $\psi_0, \ldots, \psi_s$ are a base for $\mathbb{R}_s[z]$, g must be a multiple of ψ_s, i.e. $q_1, \ldots, q_s$ are the zereos of ψ_s. The rest of the Theorem follows from [1], Thm. (5.4). $\square$

We can use (3.1) to give a new proof of a stronger form of McWilliams inequality $s(a) \geqq [t(a^{\mathrm{T}} \mathrm{Q})/2]$ ([1], Thm. (5.5)) and to treat the case of equality (not discussed in [1]).

(3.2) **Theorem:** *Let $a \in \mathbb{R}^{n+1}$ with $\mathrm{A} := \sum_{i=0}^{n} a_i \neq 0$.*

Then $2s(a) \geqq t(a^{\mathrm{T}} \mathrm{Q})$.

Moreover, in case of equality we have, with $s = s(a)$, $i_1, \ldots, i_s$ the values of $i \neq 0$ such that $a_i \neq 0$, $z_{i_1}, \ldots, z_{i_s}$ the zeroes of ψ_s and $w_1, \ldots, w_s$ the corresponding Christoffel-numbers:

$$a_{i_j} = \mathrm{A}\,|\mathrm{X}|^{-1}\, w_j z_{i_j}^{-1} \qquad (j = 1, \ldots, s)$$

and

$$a_0 = \mathrm{A}/\psi_s(0).$$

Proof: With $t := t(a^{\mathrm{T}} \mathrm{Q})$, we have

$$0 = a^{\mathrm{T}} \mathrm{Q}_k = \sum_{i=0}^{n} a_i \mathrm{Q}_k(i) = \sum_{j=0}^{s} a_{i_j} \varphi_k(z_{i_j})$$

$$(k = 1, \ldots, t)$$

and

$$a^{\mathrm{T}} \mathrm{Q}_0 = \mathrm{A}.$$

So:

$$(1) \qquad \sum_{j=0}^{s} a_{i_j} \varphi_k(z_{i_j}) = \mathrm{A}\,\delta_{k,0} = \mathrm{A}\,|\mathrm{X}|^{-1} (\varphi_k, 1)$$

for $k = 0, 1, \ldots, t$.

Now $\varphi_0, \ldots, \varphi_t$ are a base for $\mathbb{R}_t[z]$, so (1) holds also if $\varphi_k(z)$ is replaced by $z f(z)$ for some $f(z) \in \mathbb{R}_{t-1}[z]$ and we find:

$$(2) \quad \sum_{j=1}^{s} (a_{i_j} z_{i_j} \mathrm{A}^{-1} |\mathrm{X}|) f(z_{i_j}) = [f, 1] \qquad \text{for all } f(z) \in \mathbb{R}_{t-1}[z].$$

Now apply (3.1). To find the value of a_0, note that as a consequence of (1), we also have:

$$(3) \quad \sum_{j=0}^{s} a_{i_j} f(z_{i_j}) = \mathrm{A}\,|\mathrm{X}|^{-1} (f, 1) \qquad \text{for all } f(z) \in \mathrm{R}_t[z].$$

Now take:

$$f(z) = \psi_s(z)/\psi_s(0).$$

Then:

$$(f, 1) = \psi_s(0)^{-1} (\psi_0, 1) = |\mathrm{X}|\,\psi_s(0)^{-1},$$

$$f(0) = 1 \quad \text{and} \quad f(z_{i_j}) = 0 \quad (j = 1, \ldots, s),$$

hence from (3) we find:

$$a_0 = \mathrm{A}\,|\mathrm{X}|^{-1} (f, 1) = \mathrm{A}/\psi_s(0). \quad \square$$

4. The generalised Rao-Wilson bound

Let Φ be a design with support Y, degree s, maximum strength t, inner-distribution a and outer distribution matrix B, and let $b = \sum_{y \in \mathrm{Y}} \Phi(y)$ be the total number of points of Φ.

Let $y \in \mathrm{Y}$. We define s_y, the *degree relative* to y, and t_y, the *maximum strength relative* to y, as

$$s_y := s(\mathrm{B}(y)), \qquad t_y := t(\mathrm{B}(y)\mathrm{Q})$$

[where $\mathrm{B}(y)$ denotes the row of B indexed by y].

We have:

(4.1) **Lemma:** $t \leqq t_y \leqq 2s_y \leqq 2s$.

Proof: The maximum strength of Φ being t, we have $a^{\mathrm{T}} \mathrm{Q}_k = 0$ for $k = 1, \ldots, t$, or [by (1.5)] equivalently, $\mathrm{B}\mathrm{Q}_k = 0$ for $k = 1, \ldots, t$. So certainly $\mathrm{B}(y)\mathrm{Q}_k = 0$ for $k = 1, \ldots, t$, i.e. $t \leqq t_y$.

The second inequality follows from (3.2).

Finally, $a_i = 1/b \sum_{x \in \mathrm{Y}} \Phi(x)\mathrm{B}(x, i)$ so $a_i = 0$ implies $\mathrm{B}(x, i) = 0$ for all $x \in \mathrm{Y}$, hence $s_y \leqq s$. $\square$

In [1], (5.37) it was shown that $s \geqq [t/2]$ (McWilliams-inequality). The notions of maximum strength and degree relative to y in Y are new.

Now we come to the main Theorem of this paper. Let $e = [t/2]$.

(4.2) **Theorem:** (i) *For all $y \in \mathrm{Y}$, we have:*

$$(1) \qquad b \geqq \Phi(y)\,\psi_e(0) \quad \textit{and} \quad s_y \geqq e.$$

If, for some $y \in \mathrm{Y}$, either of these bounds is attained, so is the other, and in this case, t is even and ψ_e has e zeroes $z_{i_1}, \ldots, z_{i_e}$ [where $i_1, \ldots, i_e$ are the values of $i \neq 0$ such that $\mathrm{B}(y, i) \neq 0$]. Moreover,

(2) $\quad B(y, i_k) = b|X|^{-1} z_{i_k} w_k^{-1} \qquad (k=1, \ldots, e),$

where $w_1, \ldots, w_e$ are the Christoffel-numbers of ψ_e.
(ii) *For all $x, y \in Y$ with $x \neq y$ we have:*

(3) $$(b/\Phi(x) - \psi_e(0))(b/\Phi(y) - \psi_e(0)) \geqq \{\psi_e(z_u)\}^2 \quad \text{if} \quad (x, y) \in \Gamma_u.$$

Proof: From definition (2.3) and (2.1) (iii) we find:

$$E_k(x, y) = b^{-1}\Phi(x)^{1/2}\Phi(y)^{1/2} Q_k(u) \quad \text{if } (x, y) \in \Gamma_u \qquad (k=0, 1, \ldots, e),$$

and, since $Q_k(u) = \Phi_k(z_u)$,

$$E_{e+1}(x, y) = \delta_{x,y} - b^{-1}\Phi(x)^{1/2}\Phi(y)^{1/2}\psi_e(z_u) \quad \text{if } (x, y) \in \Gamma_u.$$

Now note that, by (2.4), E_{e+1} is a symmetric idempotent.

As a first consequence, we have $E_{e+1}(y, y) \geqq 0$, i.e. $b \geqq \Phi(y)\psi_e(0)$ for all $y \in Y$.

Together with (4.1), this proves (1).

Secondly, any 2×2 principal submatrix of E_{e+1} must be non-negative definite, hence must have non-negative determinant.

So for all $x, y \in Y$ with $x \neq y$, $E(x, x) E(y, y) \geqq E(x, y)^2$. This is equivalent to (3).

To prove (i), first note that if $b = \Phi(y)\psi_e(0)$ for some $y \in Y$, it follows from (3) that $\psi_e(z_u) = 0$ for all $u \neq 0$ such that $(y, x) \in \Gamma_u$ for some $x \in Y$. So if $i_1, \ldots i_{s_y}$ are the values of $i \neq 0$ such that $B(y, i) \neq 0$ then $\psi_e(z_{i_j}) = 0$ for $j = 1, \ldots, s_y$, and ψ_e has at least s_y zeroes. Since $\psi_e(z) \in \mathbb{R}_e[z]$, it follows that $s_y \leqq e$, hence as a consequence of (4.1), we have $s_y = e$.

On the other hand, suppose $s_y = e$ for some $y \in Y$. By (4.1), we then have $2s_y = t_y = t$, so t is even. Moreover, from (3.2) we find the expressions (2) for $B(y, i_k)$, together with $\Phi(y) = B(y, 0) = b/\psi_e(0)$, i.e. $b = \Phi(y)\psi_e(0)$. □

(4.3) *Remark:* (4.2) (i) Generalises [1], Thm. (5.21) and (5.22), and provides new proves of these Theorems. Moreover, (4.2) (ii) generalises a result of Connor for classical 2-designs (*see* [5] and also [3]).

The following Theorem should be compared to [1], Thm. (5.24):

(4.4) **Theorem:** *If $t \geqq s$ then Φ is constant on* Y.

Proof: As in the proof of (4.1), we find:

$$B(y) Q_k = 0 \qquad (k=1, \ldots, t, y \in Y).$$

Since also $B(y) Q_0 = b$, we can write, if $i_0 = 0, i_1, \ldots, i_s$ are the values of i such that $a_i \neq 0$:

$$\sum_{j=0}^{s} B(y, i_j)\varphi_k(z_{i_j}) = b\,\delta_{k,0} = b|X|^{-1}(\varphi_k, 1) \qquad (k=0, \ldots, t, y \in Y).$$

Now $\varphi_0, \ldots, \varphi_t$ are a base for $\mathbb{R}_t[z]$, so it follows that:

(4) $$\sum_{j=0}^{s} B(y, i_j) f(z_{i_j}) = b|X|^{-1}(f, 1) \qquad [f(z) \in \mathbb{R}_t[z], y \in Y].$$

With f_k defined by:

$$f_k(z) := \prod_{j \neq k} \frac{(z - z_{i_j})}{(z_{i_k} - z_{i_j})},$$

f_k has degree $s \leqq t$ and $f_k(z_{i_j}) = \delta_{k,j}$, and we find from (4): $B(y, i_k) = b|X|^{-1}(f_k, 1)$, independent of $y \in Y$.
In particular, $\Phi(y) = B(y, 0)$ is constant on Y. □

Conclusions

A generalisation is proved of the Rao-Wilson bound and the dual of Lloyds Theorem in case of equality for t-designs with repeated points in Q-polynomial schemes.

Moreover, a stronger form of McWilliams inequality is derived. This shows in particular that the fact that the inner-distribution of perfect codes and of tight designs is determined by the parameters of the scheme is a direct consequence of equality in the McWilliams bound.

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