Representer theorems for convex regularization

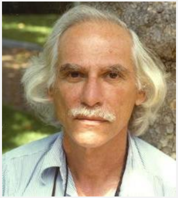
C. Boyer, Y. de Castro, A. Chambolle, V. Duval, A. Flinth, F. de Gournay, P. Weiss

Beijing, 21/04/2018

Introduction

The (older) heroes of this talk

Constantin Caratheodory (1873-1950)



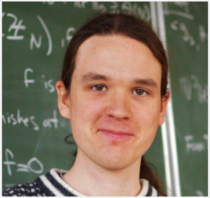
Lester Dubins (1920-2010)

Victor Klee (1925-2007)

And S.I. Zuhovickiĭ, G. Choquet, S. Fisher, J. Jerome...

Introduction

The (younger) heroes of this talk

Axel Flinth (1992-)

Vincent Duval (1983-)

Frédéric de Gournay (1980-)

And D. Donoho, V. Chandrasekaran etal, R. Chan,...

Introduction

Inverse problems

Let $u \in \mathcal{B}$, denote a signal from a vector space $\mathcal{B}$ (finite or infinite).
We are given a finite number m of corrupted linear measurements:

$$y = P(Au),$$

where

- $A : \mathcal{B} \to \mathbb{R}^m$ is defined by

$$(Au)_i = \langle a_i, u \rangle, a_i \in \mathcal{B}^*$$

- $P : \mathbb{R}^m \to \mathbb{R}^m$ is a perturbation operator (e.g. quantization, additive noise, modulus...).

Problem

How can we retrieve an approximation $\hat{u}$ of u knowing y and A?

Introduction

Example 1: Photography

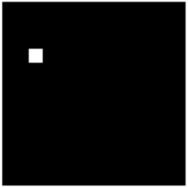
On a conventional camera:

$$a_i(\cdot) = h(\cdot - x_i)$$

where h is a function localized around 0 and x_i denotes a pixel center.

$$y_m = \Bigg\langle$$

,



$$\Bigg\rangle$$

Introduction

Example 2: Tomography

In tomography a_i allows measuring line integrals.

$y_m = \Big\langle$  ,  $\Big\rangle$

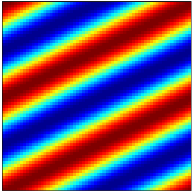
Introduction

Example 3: MRI

In MRI the functions a_i are complex exponentials.

$y_m = \Big\langle$

,



$\Big\rangle$

Introduction - Quadratic regularization

A critical issue

Regularization is critical whenever $\dim(\mathcal{B}) > m$.

Introduction - Quadratic regularization

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Tikhonov regularization (before 1943)

When $\mathcal{B}$ is a Hilbert space, we can solve:

$$\inf_{u \in \mathcal{B}} \frac{1}{2}\|Au - y\|_2^2 + \|Lu\|_{L^2}^2,$$

where $L : \mathcal{B} \to L^2$ is a linear operator (e.g. the derivative)

- ✔ Solutions given by linear systems.
- ✔ Sometimes solution of a finite dimensional problem yields an infinite dimensional solution (RKHS).
- × Typically restricts $\mathcal{B}$ to Hilbert spaces such as $W^{n,2}$.

Introduction - Quadratic regularization

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- ✔ Solutions given by linear systems.
- ✔ Sometimes solution of a finite dimensional problem yields an infinite dimensional solution (RKHS).
- × Typically restricts $\mathcal{B}$ to Hilbert spaces such as $W^{n,2}$.
- × Solutions live in a fixed subspace that depends on A and L only:

$$u^\star = \sum_{i=1}^{m} \alpha_i \psi_i + u_K, \quad \text{where } u_K \in \ker(L). \tag{1}$$

A first representer theorem.

Introduction - More recent approaches

Analysis formulation (before 1973)

$$\inf_{u\in\mathcal{B}} f_y(Au) + \|Lu\|_{\mathcal{M}},$$

- $L : \mathcal{B} \to \mathcal{M}$ is a linear operator (e.g. the derivative).
- $\mathcal{M}$ is the space of Radon measures.
- $f_y : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is a data fitting term.

Synthesis formulation (before 1973)

$$\inf_{\mu\in\mathcal{M}} f_y(AD\mu) + \|\mu\|_{\mathcal{M}},$$

where $D : \mathcal{M} \to \mathcal{B}$ is a linear operator called dictionary.

The estimate of $\hat{u}$ is given by $\hat{u} = D\hat{\mu}$.

S.D. Fisher and J.W. Jerome.
Spline solutions to l1 extremal problems in one and several variables.
Journal of Approximation Theory, 13(1):73–83, 1975.

Introduction - More recent approaches

Other popular examples

Nonnegative least squares:

$$\inf_{u\in B, u\geq 0} \frac{1}{2}\|Au-y\|_2^2.$$

Nuclear norm minimization:

$$\inf_{u\in\mathcal{B}, Au=y} \|u\|_*.$$

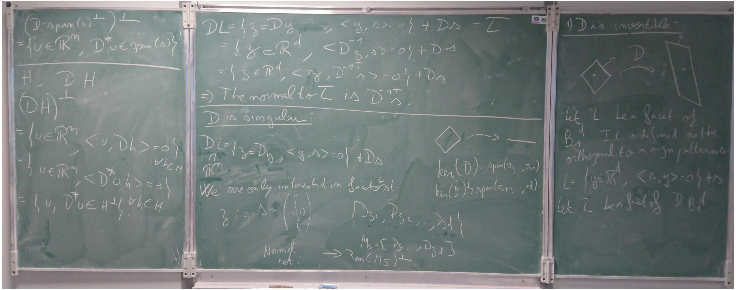
Plenty of such examples scattered in the literature.

The question tackled today

Can we derive representer theorems for problems of the form:

$$\inf_{u\in\mathcal{B}} f_y(Au)+R(u), \text{ where } R \text{ is convex?}$$

Part I: The main theoretical results



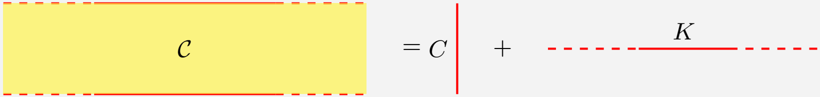
Preliminaries

Decomposition of a convex set

Let $\mathcal{C}$ denote a (linearly closed) convex set in $\mathcal{B}$.

We can decompose $\mathcal{C} = K + C$, where

- $K = \mathrm{Lin}(\mathcal{C})$ is the lineality space of $\mathcal{C}$,
- C is a linearly closed set that contains no line.



Convex gauge

The gauge of $\mathcal{C}$ is defined by:

$$R_{\mathcal{C}}(u) = \inf_{\lambda \geq 0, u \in \lambda \mathcal{C}} \lambda$$

Preliminaries

Our setting

I will focus on the properties of the minimizers of:

$$\inf_{u \in \mathcal{B}} f_y(Au) + R_{\mathcal{C}}(u).$$

where $f_y : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is either:

- a convex closed function.
- an arbitrary (nonconvex function).

We assume that the set of minimizers $\hat{U}$ is non empty.

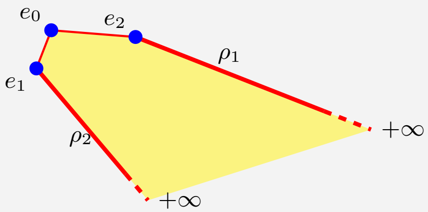
Preliminaries

Carathéodory - Klee (1957)

Let C denote a linearly closed convex set that contains no line in dimension m.

Then any point $u \in C$ can be expressed either as:

- A convex combination of $m+1$ points in $\mathrm{Ext}(C)$.
- A convex combination of m points in $\mathrm{Ext}(C) \cup \mathrm{Ray}(C)$.



V. Klee.
Extremal structure of convex sets.
Archiv der Mathematik, 8(3):234–240, August 1957.

Preliminaries

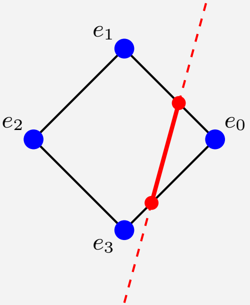
Dubins - Klee (1963)

Let C denote a linearly closed convex set that contains no line.
Let H denote an affine space of co-dimension m.

Then the extreme points and extreme rays of $C \cap H$ can be expressed as:

- A convex combination of $m + 1 + j$ points in $\text{Ext}(C)$.
- A convex combination of $m + j$ points in $\text{Ext}(C) \cup \text{Ray}(C)$.

Where $j = 0$ for the extreme points and $j = 1$ for the extreme rays.



Main results

A representer theorem: the nonconvex case (New result)

Consider the problem:

$$t^\star = \inf_{u \in \mathcal{B}} f_y(Au) + R_{\mathcal{C}}(u),$$

where f_y is an arbitrary function. Assume that at least one solution exists.

Then there exists a solution $\hat{u}$ of the form:

$$\hat{u} = \sum_{i=1}^{m+z} \alpha_i \psi_i + u_K,$$

where

- $u_K \in \mathrm{Lin}(\mathcal{C})$.
- $\psi_i \in \mathrm{Ext}(C) \cup \mathrm{Ray}(C)$ are the atoms of C.
- $z \leq 1_{t^\star = 0} - \dim(AK)$.

The bound is tight.

Main results

A representer theorem: the convex case (New result)

Consider the problem:

$$t^\star = \inf_{u \in \mathcal{B}} f_y(Au) + R_{\mathcal{C}}(u),$$

where f_y is either strictly convex or the indicator of a convex, linearly closed set. Assume that at least one solution exists.

Then the extreme points and rays of the solution set $\hat{U}$ are of the form:

$$\hat{u} = \sum_{i=1}^{m+z} \alpha_i \psi_i + u_K,$$

where

- $u_K \in \mathrm{Lin}(\mathcal{C})$.
- $\psi_i \in \mathrm{Ext}(C) \cup \mathrm{Ray}(C)$ are the atoms of C.
- $z \leq 1_{t^\star=0} + j + -\dim(AK)$, with $j = 0$ for extreme points and $j = 1$ for extreme rays.

Main results

The (rough) proof

Let $u^\star$ denote a solution and $t^\star = R_{\mathcal{C}}(u^\star)$. Consider the problem:

$$\inf_{u \in \mathcal{B}, Au = Au^\star} R_{\mathcal{C}}(u)$$

Any solution $\hat{u}$ is a solution of the original problem and satisfies $R_{\mathcal{C}}(\hat{u}) = t^\star$.
So $\hat{U} = H \cap D$, where:

$$H = \{u \in \mathcal{B}, Au = Au^\star\}$$

and

$$D = \{u \in C, R_{\mathcal{C}}(u) \leq t^\star\}.$$

Applying Klee's theorem (on $H \cap D$ quotiented by K), we get a complete description of this subset.
We can gain 1 point since the solutions live on the boundary of $\mathcal{C}$.

Part II: Examples of applications

Applications

ℓ_1 and total variation minimization

Consider the problems:

$$\inf_{u \in \mathbb{R}^n} f_y(Au) + \|u\|_1$$

or

$$\inf_{u \in \mathcal{M}} f_y(Au) + \|u\|_{\mathcal{M}}$$

There is at least one solution are m sparse:

$$\hat{u} = \sum_{i=1}^{m} \alpha_i \delta_{z_i}.$$

S.C. Chen, D. Donoho, and M. Saunders.
Atomic decomposition by basis pursuit.
SIAM review, 43(1):129–159, 2001.

D.L. Donoho.
Compressed sensing.
IEEE T. Inf. Theory, 52(4):1289–1306, 2006.

E. Candès and C. Fernandez-Granda.
Towards a mathematical theory of super-resolution.
Communications on Pure and Applied Mathematics, 67(6):906–956, 2014.

Applications

Nonnegative constraints

Consider the problem:

$$\min_{u \in \mathbb{R}^n_+} \frac{1}{2} \|Au - y\|_2^2$$

Then the extreme points and rays of the solution set are m sparse.

Don't use ℓ^1 when looking for sparse nonnegative signals!

D. Donoho and J. Tanner.
Sparse nonnegative solution of underdetermined linear equations by linear programming.
P. Nat. Acad. Sci. USA, 102(27):9446–9451, 2005.

A. Eftekhari, J. Tanner, A. Thompson, B. Toader, and H. Tyagi.
Sparse non-negative super-resolution-simplified and stabilised.
arXiv preprint arXiv:1804.01490, 2018.

Applications

Analysis priors - finite dimension

Let $L \in \mathbb{R}^{m \times n}$ denote a linear mapping.
Consider the problem:

$$\min_{u \in \mathbb{R}^n} f_y(Au) + \|Lu\|_1.$$

Then:

- If L is surjective, at least one solution can be written as:

$$\hat{u} = \sum_{i=1}^{m} \alpha_i L^+ \delta_{z_i} + u_K, u_K \in \ker(L).$$

- If L is not surjective, then there is a combinatorial explosion of the extreme points:

$$\#\mathrm{Ext}(\{u \in \mathbb{R}^n, \|Lu\|_1 \leq 1\}) \leq 2^{m-n+1} C_m^{m-n+1}$$

  Finding the vertices is the convex hull problem.

Applications

Analysis priors - infinite dimension (Old and new results)

Let $L : \mathcal{B} \to \mathcal{M}$ denote a linear and surjective mapping (plus some technical assumptions), where

$$\mathcal{B} = \{u \in \mathcal{D}', Lu \in \mathcal{M}, \|u\|_K < \infty\}.$$

Consider the problem:

$$\inf_{u \in \mathcal{B}} f_y(Au) + \|Lu\|_{\mathcal{M}}.$$

Then at least one solution is of the form:

$$\hat{u} = \sum_{i=1}^{m} \alpha_i L^+(\delta_{z_i}) + u_K.$$

S.D. Fisher and J.W. Jerome.
Spline solutions to l1 extremal problems in one and several variables.
Journal of Approximation Theory, 13(1):73–83, 1975.

M. Unser, J. Fageot, and John P. Ward.
Splines are universal solutions of linear inverse problems with generalized tv regularization.
SIAM Review, 59(4):769–793, 2017.

A. Flinth and P. Weiss.
Exact solutions of infinite dimensional total-variation regularized problems.
arXiv preprint arXiv:1708.02157, 2017.

Applications

Analysis priors - Biharmonic approximation (New result)

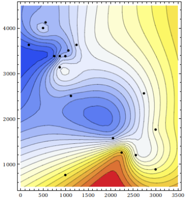
Solve

$$\inf_{u\in\mathcal{B}} \frac{1}{2}\sum_{i=1}^{m}(u(x_i)-y_i)^2 + \|\Delta\Delta u\|_{\mathcal{M}}.$$

Letting $\psi(x) = \|x\|^2\log(\|x\|)$, we get a solution of the form:

$$\hat{u} = \sum_{i=1}^{m}\alpha_i\psi(\cdot - z_i) + u_K,$$

is a polyharmonic spline, with u_K a polynomial of degree 1.



Polyharmonic splines are used for data interpolation

Applications

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is a polyharmonic spline, with u_K a polynomial of degree 1.

The traditional approach

Usually, polyharmonic splines are appearing in the frame of RKHS.

$$\inf_{u\in H^2(\mathbb{R}^2)} \frac{1}{2}\sum_{i=1}^{m}(u(x_i)-y_i)^2 + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}.$$

Applications

Total gradient variation (New result)

Consider the following problem:

$$\inf_{u \in BV(\mathbb{R}^d)} f_y(Ax) + \|Du\|_{\mathcal{M}},$$

then there exists a solution of the form:

$$\hat{u} = \sum_{i=1}^{m} \alpha_i \psi_i + c,$$

where c is a constant and

$$\psi_i = \mathbb{1}_{\omega_i}, \text{ where } \omega_i \text{ is a simple set.}$$

W.H. Fleming.
Functions with generalized gradient and generalized surfaces.
Annali di Matematica Pura ed Applicata, 44(1):93–103, 1957.

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Connected components of sets of finite perimeter and applications to image processing.
Journal of the European Mathematical Society, 3(1):39–92, 2001.

Applications

Other applications...

- Nuclear norm minimization ⇒ low rank.
- Linear, semi-definite and conic programming ⇒ sparse, low rank.
- Optimal transport ⇒ permutation matrices.
- Rank sparsity ball ⇒ low rank and sparse.
- ...

Some notes on computing

Representer theorems allow solving infinite dimensional problem exactly!

E. Candès and C. Fernandez-Granda.
Towards a mathematical theory of super-resolution.
Communications on Pure and Applied Mathematics, 67(6):906–956, 2014.

V. Duval and G. Peyré.
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Some references

Thank you very much!

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