1. Physics of imaging
Unconventional imaging and co-design
Part II: Image formation, Mathematical model

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june 2021

Outline

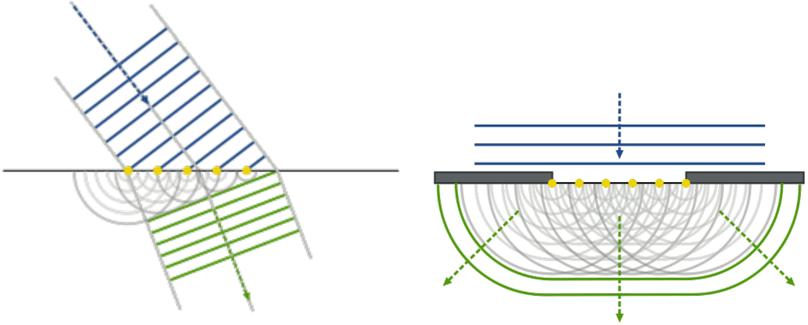
1. Coherent imaging
   - Light propagation model
     - Huygens-Fresnel principle
     - Rayleigh Sommerfeld model/ Fresnel model /Fraunhofer model
     - Simulation of light propagation in free space : summary
   - Propagation models properties
   - Implementation of the free space propagation
   - Image formation model
     - Image formation by a lens
     - Image formation : holography
     - Image formation : Phase masks and aberrations
2. Incoherent imaging
   - Incoherent PSF

Coherent imaging

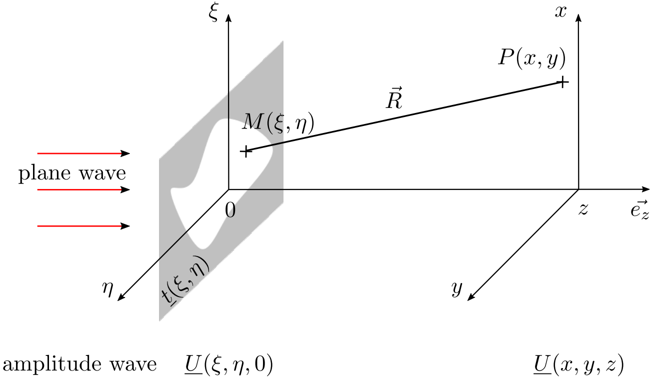
Light propagation model

Huygens-Fresnel principle (scalar theory)

The Huygens-Fresnel principle states that each point of a medium (disturbed by passing wave) becomes source of disturbance which propagates from this point in all directions indiscriminately. (Indeed, when a uniform medium is disturbed at some point then due to directional symmetry this disturbance propagates in all directions equally and without any path/direction discrimination). The interference (=addition) of all disturbances then results in a certain amplitude of detected wave.

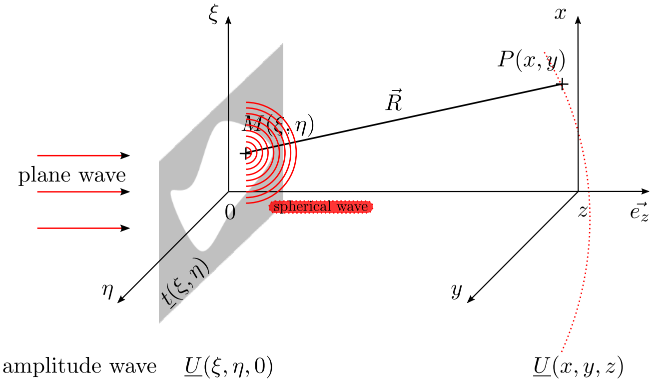


Simulation of light propagation in free space



$$R = ||\vec{MP}|| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$$

Simulation of light propagation in free space



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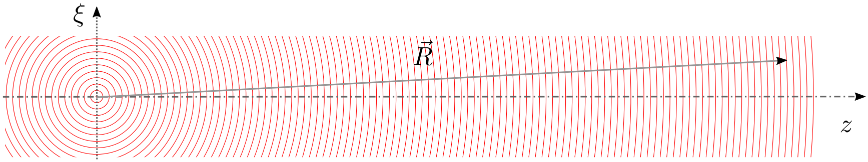


Spherical | Paraboloidal | Planar

Rayleigh Sommerfeld > Fresnel > Fraunhofer

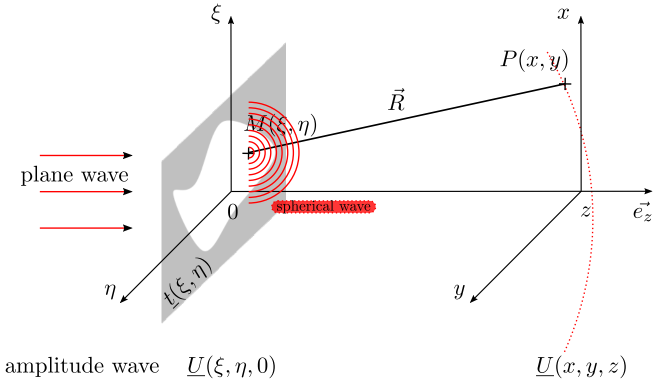
$$kR = k\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2} \quad (*)$$

$$kR \simeq kz + k\frac{(x-\xi)^2 + (y-\eta)^2}{2z}$$

$$kR \simeq kz + k\frac{x^2 + y^2}{2z} - k\frac{2x\xi + 2y\eta}{2z}$$

(*) with $k = n\frac{2\pi}{\lambda}$, n is the refractive index of the medium.

Rayleigh Sommerfeld model



$$\underline{U}(x,y,z) = \frac{1}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}(\xi,\eta,0)\frac{e^{jkR}}{R}\cos\theta d\xi d\eta$$

$R = ||\vec{MP}|| = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2}$ and $\cos\theta = \cos(\vec{e_z},\vec{R}) = \frac{z}{R}$

* Goodman, "Introduction to Fourier optics", Roberts and Company Publishers, 2005

Rayleigh Sommerfeld model

$$\underline{U}(x, y, z) = \frac{1}{j\lambda} \int\int_{-\infty}^{+\infty} \underline{U}(\xi, \eta, 0) \frac{e^{jkR}}{R} \cos\theta d\xi d\eta$$

- Replace $\cos(\theta)$ by its expression
- Replace R by its expression
- Use the convolution product definition to simplify the expression

Rayleigh Sommerfeld model

$$\underline{U}(x,y,z) = \frac{1}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}(\xi,\eta,0)\frac{e^{jkR}}{R}\cos\theta \, d\xi d\eta$$

- Replace $\cos(\theta)$ by its expression
- Replace R by its expression
- Use the convolution product definition to simplify the expression

$$\underline{U}(x,y,z) = \frac{z}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}(\xi,\eta,0)\frac{e^{jkr}}{R^2} d\xi d\eta$$

$$\underline{U}(x,y,z) = \frac{z}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}(\xi,\eta,0)\frac{ejk\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}{(x-\xi)^2+(y-\eta)^2+z^2} d\xi d\eta$$

Reminder convolution product :

$$f(x,y) = \int\int_{-\infty}^{+\infty} g(\xi,\eta).h(x-\xi,y-\eta)d\xi d\eta = g \underset{x,y}{*} h$$

$$\boxed{\underline{U}_z(x,y) = \underline{U} \underset{x,y}{*} \underline{h}_z \quad \text{with} \quad \underline{h}_z(\xi,\eta) = \frac{z}{j\lambda}\frac{e^{jk\sqrt{\xi^2+\eta^2+z^2}}}{\xi^2+\eta^2+z^2}}$$

Rayleigh Sommerfeld model

In space domain :

$$\underline{U}_z(x,y) = \underline{U} \underset{x,y}{*} \underline{h}_z \quad \text{with} \quad \underline{h}_z(\xi,\eta) = \frac{z}{j\lambda} \frac{e^{jk\sqrt{\xi^2+\eta^2+z^2}}}{\xi^2+\eta^2+z^2}$$

$\underline{h}_z$ is called :

- the propagation kernel
- the impulse response
- the PSF (Point Spread Function)

In Fourier domain (Angular Spectrum) :

$$\tilde{\underline{U}}_z(\mu_x,\mu_y) = \tilde{\underline{U}}_0(\mu_x,\mu_y).\tilde{\underline{h}}_z(\mu_x,\mu_y)$$

$$\text{with } \tilde{\underline{h}}_z(\mu_x,\mu_y) = e^{jzk\sqrt{1-\frac{(2\pi\mu_x)^2}{k^2}-\frac{(2\pi\mu_y)^2}{k^2}}}$$

$\tilde{\underline{h}}_z$ is called :

- the Transfer Function
- the amplitude Transfer Function

Fresnel approximation

$$\underline{U}_z(x,y) = \frac{z}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)\frac{e^{jk\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}}{(x-\xi)^2+(y-\eta)^2+z^2}d\xi d\eta$$

For large z, $\cos(\theta) \simeq 1$ and the phase expression of the spherical wave kR can be simplified in (2.1) using a Taylor's serie expansion of order 1.

$$kR = kz\left(1 + \frac{(x-\xi)^2}{z^2} + \frac{(y-\eta)^2}{z^2}\right)^{\frac{1}{2}} \simeq kz\left(1 + \frac{(x-\xi)^2}{2z^2} + \frac{(y-\eta)^2}{2z^2}\right)$$

- Replace $\cos(\theta)$ by 1 and kR by its approximation
- Use the convolution product definition to simplify the expression

Fresnel approximation

$$\underline{U}_z(x,y) = \frac{z}{j\lambda}\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)\frac{e^{jk\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}}{(x-\xi)^2+(y-\eta)^2+z^2}d\xi d\eta$$

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- Replace $\cos(\theta)$ by 1 and kR by its approximation
- Use the convolution product definition to simplify the expression

$$\underline{U}_z(x,y) = \frac{1}{j\lambda z}\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)e^{jk\left(z+\frac{(x-\xi)^2}{2z}+\frac{(y-\eta)^2}{2z}\right)}d\xi d\eta$$

Note : In the denominator R has been replaced by z.
This simplification cannot be done in the phase term because a small change in R in the phase term leads to a big change in the phase (e.g., $\delta R = \lambda/2$, $\delta\phi = \pi$).

Fresnel approximation

$$\underline{U}_z(x,y) = \frac{1}{j\lambda z}\int\int_{-\infty}^{+\infty} \underline{U}_0(\xi,\eta) e^{jk\left(z+\frac{(x-\xi)^2}{2z}+\frac{(y-\eta)^2}{2z}\right)} d\xi d\eta$$

Using the convolution product :

$$\boxed{\underline{U}_z(x,y) = \underline{U} \underset{x,y}{*} \underline{h}_z^{Fr} \quad \text{with} \quad \underline{h}_z^{Fr}(\xi,\eta) = \frac{1}{j\lambda z} e^{jkz} e^{jk\frac{\xi^2+\eta^2}{2z}}}$$

Fresnel assumption is valid if the second order term in the Taylor series of R is negligeable : $z^3 > 10.\frac{\pi}{4\lambda}\{[(x-\xi)^2+(y-\eta)^2]^2\}_{max} = z^{RS-Fr}$
For a diffractive area and a observation area squared of width W :
$z^3 > 10.\frac{\pi}{\lambda}W^4$
In most many practical cases, a less restrictive condition can be applied : $z \gg 64\pi\lambda$
* Goodman, "Introduction to Fourier optics", Roberts and Company Publishers, 2005

Fraunhofer approximation

$$\underline{U}_z(x,y) = \frac{1}{j\lambda z}\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)e^{jk\left(z+\frac{(x-\xi)^2}{2z}+\frac{(y-\eta)^2}{2z}\right)}d\xi d\eta$$

- Expand $(x-\xi)^2$ and $(y-\eta)^2$ in the phase and simplify the expression
- simplify the expression by neglecting the term $k\frac{(\xi^2+\eta^2)}{2z}$

Fraunhofer approximation

$$\underline{U}_z(x,y) \simeq \frac{e^{jkz}e^{jk\frac{(x^2+y^2)}{2z}}}{j\lambda z}\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)e^{-jk(\frac{x}{z}\xi+\frac{y}{z}\eta)}\cancel{e^{jk\frac{(\xi^2+\eta^2)}{2z}}}d\xi d\eta$$

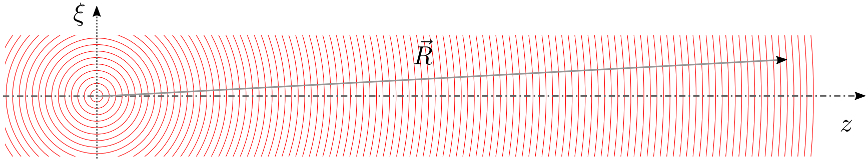
$$\underline{U}_z(x,y) \simeq \frac{e^{jkz}e^{jk\frac{(x^2+y^2)}{2z}}}{j\lambda z}\underbrace{\int\int_{-\infty}^{+\infty}\underline{U}_0(\xi,\eta)e^{-j2\pi(\frac{x}{\lambda z}\xi+\frac{y}{\lambda z}\eta)}d\xi d\eta}_{\text{Fourier Transform } \mathcal{F}}$$

$$\boxed{\underline{U}_z(x,y) \simeq \frac{e^{jkz}e^{jk\frac{(x^2+y^2)}{2z}}}{j\lambda z}\mathcal{F}_{\mu_x,\mu_y}[\underline{U}_0(\xi,\eta)] \text{ with } \mu_x = \frac{x}{\lambda z} \text{ and } \mu_y = \frac{y}{\lambda z}}$$

Fraunhofer assumption is valid if :

- Fresnel approximation is valid, i.e. $z^3 > 10.\frac{\pi}{4\lambda}\{[(x-\xi)^2+(y-\eta)^2]^2\}_{max}$
- $\pi\left(\frac{\xi^2+\eta^2}{\lambda z}\right)_{max} \ll 1$
  If D is the width of a circular aperture in the transmittance plane :
  $z \gg \pi\frac{D^2}{4\lambda}$

Simulation of light propagation in free space : summary



Spherical	Paraboloidal	Planar
Rayleigh Sommerfeld >	Fresnel >	Fraunhofer
$\underline{U}_z(x,y) = \underline{U}_0 \underset{x,y}{*} \underline{h}_z^{RS}$	$\underline{U}_z(x,y) \simeq \underline{U} \underset{x,y}{*} \underline{h}_z^{Fr}$	$\underline{U}_z(x,y) \simeq\propto \tilde{\underline{U}}(\frac{x}{\lambda z}, \frac{y}{\lambda z}, 0)$
$\underline{h}_z^{RS}(\xi,\eta) = \frac{z}{j\lambda} \frac{e^{jk\sqrt{\xi^2+\eta^2+z^2}}}{\xi^2+\eta^2+z^2}$	$\underline{h}_z^{Fr}(\xi,\eta) = \frac{1}{j\lambda z} e^{jkz} e^{jk\frac{\xi^2+\eta^2}{2z}}$	$e^{-j2\pi(\mu_x\xi+\mu_y\eta)}$
Interferences of Spherical waves	Interferences of Paraboloidal waves	Interferences of Plane waves

Note 1 : Fraunhofer propagation is called "far field" propagation whereas Rayleigh-Sommerfeld and Fresnel propagation are called "near field propagations".
Note 2 : For $z \lesssim lambda$, scalar approximation is no more valid

Propagation models properties

Propagation models properties

The propagation operators have properties linked with physical properties of light propagation.

In the case of Fresnel propagation :
the Fresnel kernel, which is a chirp function, has a Gaussian form [1], its Fourier Transform (FT) is also a Gaussian :

$$\underline{h}_z(\xi,\eta) = \frac{e^{ikz}}{i\lambda z} e^{\frac{i\pi}{\lambda z}(\xi^2+\eta^2)} \quad \leftrightarrow \quad \begin{aligned} \mathcal{F}_{\mu_x,\mu_y}(\underline{h}_z) &= \tilde{\underline{h}}_z(\mu_x,\mu_y) \\ &= e^{ikz} e^{-i\pi\lambda z(\mu_x^2+\mu_y^2)} \end{aligned}$$

1. FT of a Gaussian function : $f(x) = e^{-\frac{x^2}{a^2}} \leftrightarrow \widetilde{f}(\mu_x) = a\sqrt{\pi}e^{-a^2(\pi\mu_x)^2}$

Propagation models properties

Additivity = Propagation $\underline{h}_{z_1} * \underline{h}_{z_2} = \underline{h}_{z_1+z_2}$	$\leftrightarrow$	$\widetilde{\underline{h}}_{z_1}.\widetilde{\underline{h}}_{z_2} = \widetilde{\underline{h}}_{z_1+z_2}$
Inverse = Inverse wave propagation $\underline{h}_z * \underline{h}_{-z} = \delta$	$\leftrightarrow$	$\widetilde{\underline{h}}_z.\widetilde{\underline{h}}_{-z} = 1$
Neutral element/Identity element =propagation of a plane wave $\underline{A} * \underline{h}_z = \underline{A}$	$\leftrightarrow$	$\underline{A}\delta.\widetilde{\underline{h}}_z = \underline{A}\delta$

Comment 1 : Fresnel transform is redundant, all information is in each plane z.
Comment 2 : These properties are true in a perfect analog world, with an infinite number of pixels and pixel size equal to zero.

PhD Thesis of Loic Denis, "Traitement et analyse quantitative d'hologrammes

Implementation of the free space propagation

Implementation of the free space propagation

Let us consider the near field propagation. Whatever is the model, RS's one or Fresnel's one, the propagation can be expressed in a convolution form.
$\underline{U}_z(x, y) = \underline{U}_0 \underset{x,y}{*} \underline{h}_z$
Using the convolution theorem, Fourier Transforms $\mathcal{F}$ can be used to compute the convolution :

$$\underline{U}_z = \mathcal{F}^{-1}[\mathcal{F}(\underline{U}_0).\mathcal{F}(\underline{h}_z)] \tag{1}$$

$\mathcal{F}(\underline{h}_z) = \underline{\widetilde{h}_z}$ can be computed analytically in Frequency domain :

$$\underline{U}_z = \mathcal{F}^{-1}[\mathcal{F}(\underline{U}_0).\underline{\widetilde{h}_z}] \tag{2}$$

The complexity is less in eq(2)...

Sampling Issues

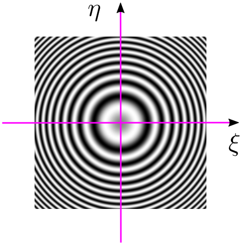
Several points have to be considered when implementing a propagation operator :

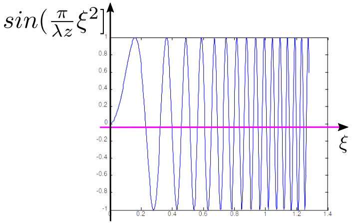
- Physical Issue
  validity of the optical approximation
- Numerical Issue
  Sampling of the Fresnel function
  Kernel separability
  Borders effects

Sampling of the propagation kernel

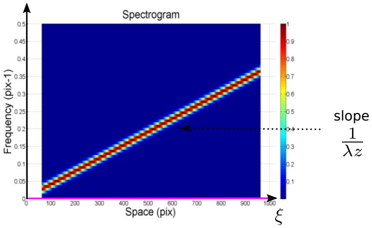
The spatial frequencies of the propagation kernels increase with the radial distance

In the case of Fresnel kernel :

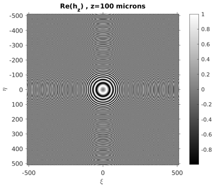


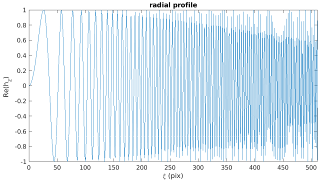


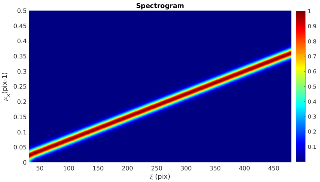
Spectrogram :
gives the frequency as a function of the space variable



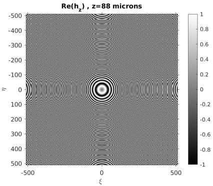
Sampling of the propagation kernel

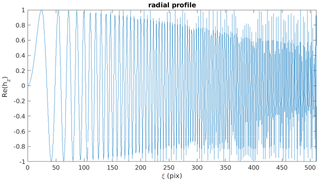


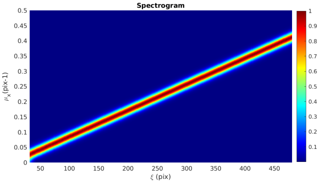




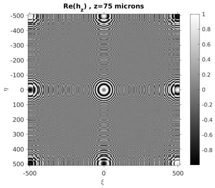
Sampling of the propagation kernel

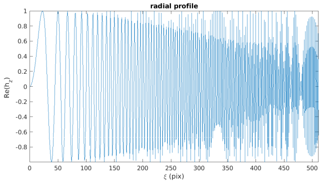


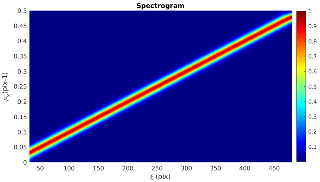




Sampling of the propagation kernel







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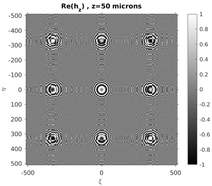

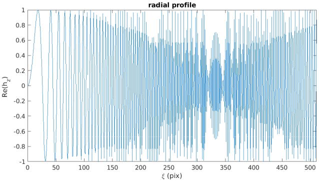

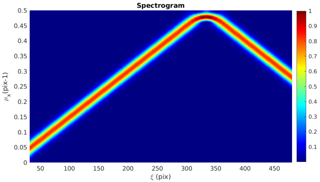

Sampling of the propagation kernel

The low threshold value of z below which the kernel is badly sampled can be estimated by considering the Shannon-Nyquist theorem

The instantaneous frequency f_{inst} of the signal should be lower than the Nyquist-Shannon frequency (equal to $\frac{1}{2p_p}$ where p_p is the pixel pitch of the digital sensor).
Reminder : $fx_{inst} = \frac{1}{2\pi}\frac{\partial\phi}{\partial x}$ where ϕ is the signal phase.

Sampling of the propagation kernel

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In the case of the Fresnel kernel (implementation : eq(1) slide 23)
$\underline{h}_z^{Fr}(x, y) = \frac{1}{j\lambda z}e^{jkz}e^{jk\frac{x^2+y^2}{2z}}$, the phase is $\phi(x, y) = -\pi/2 + kz + k\frac{x^2+y^2}{2z}$

$fx_{inst}^{max} = \frac{x^{max}}{\lambda . z} < \frac{1}{2p_p}$ thus $z > \frac{2x^{max}p_p}{\lambda}$
As $x^{max} = p_p N/2$ with N the columns number on the sensor

$$\boxed{z > z_{\text{lim sampling}}^{Fr} = \frac{Np_p^2}{\lambda}}$$

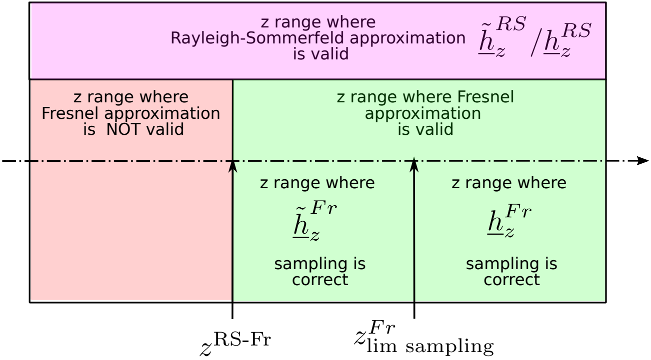
Sampling of the transfer function

In frequency domain (implementation : eq(2) slide 23) the Fourier Transform of the Fresnel kernel is :
$\mathcal{F}_{\mu_x,\mu_y}(\underline{h}_z) = \widetilde{\underline{h}}_z(\mu_x,\mu_y) = e^{jkz}e^{-j\pi\lambda z(\mu_x^2+\mu_y^2)}$

$fx_{inst}^{max} = \mu_x^{max}\lambda.z < \frac{1}{2(1/Np_p)}$ thus $z < \frac{Np_p}{2\mu_x^{max}\lambda}$
$\mu_x^{max} = \frac{1}{2p_p}$

$$\boxed{z < \frac{Np_p^2}{\lambda} = z^{Fr}_{\text{lim sampling}}}$$

Sampling and propagation summary



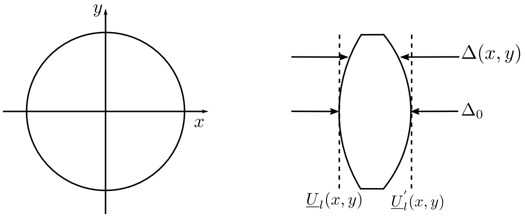
$$z^{\mathrm{RS-Fr}} = 10.\frac{\pi}{4\lambda}\{[(x-\xi)^2 + (y-\eta)^2]^2\}_{max}$$

$$z^{\mathrm{Fr}}_{\text{lim sampling}} = \frac{N p_p^2}{\lambda}$$

Image formation model

Image formation by a lens

The model of **thin lens** assumes that a lens simply **delays** the incident wavefront by an amount proportional to the thickness of the lens $\Delta(x, y)$.
Phase shift : $\phi(x, y) = n_l k_0 \Delta(x, y) + k_0[\Delta_0 - \Delta(x, y)]$ (propagation in air and n_l the refractive index of the lens material)



The transmittance of the thin lens is : $\underline{t}_l(x, y) = e^{jk_0\Delta_0} e^{jk_0(n_l-1)\Delta(x,y)}$

The complexe wave behind the lens is : $\boxed{\underline{U}_l^{'}(x, y) = \underline{t}_l(x, y)\underline{U}_l(x, y)}$

Thin lens model

In paraxial approximation (rays make a small angle θ to the optical axis) :

$\Delta(x, y) = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

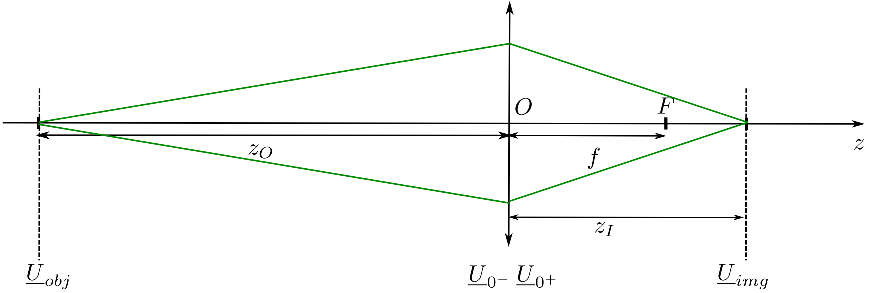
$\underline{t}_l(x, y) = e^{jk_0 n_l \Delta_0} e^{jk_0(n_l-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$

Keeping only the phase term that depends on (x, y) and introducing the lens focal length f :

$$\boxed{\underline{t}_l(x, y) = e^{-jk_0\frac{x^2+y^2}{2f}}} \text{ with } f = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

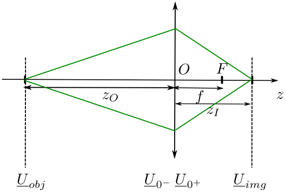
Thin lens model

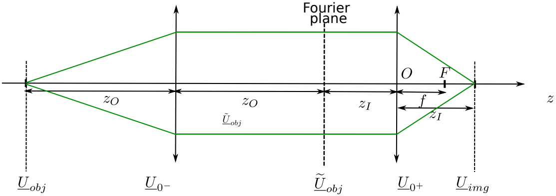
Application : Use the Fresnel propagation and the lens model to find the thin lens formula.



Thin lens model : Numerical aperture

The thin lens model is equivalent to an optical setup that includes 2 thin lens of focal length z_0 and z_I and same focal point.





$$\underline{U}^{f}_{img}(x,y) = \mathcal{F}^{-1}_{x,y}\left(\tilde{\underline{U}}_{img}(\mu_x,\mu_y).P(\tfrac{x}{\lambda z_I},\tfrac{y}{\lambda z_I})\right)$$

Image formation : holography

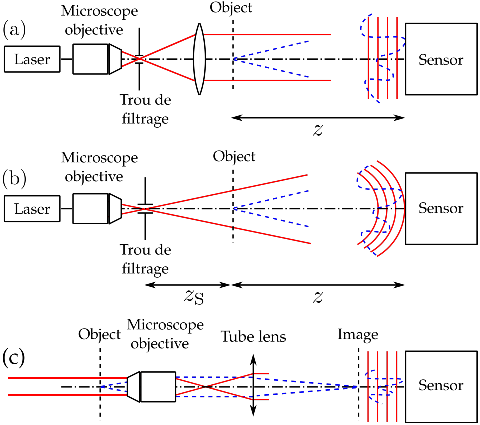
Inline Holographic setups

Most simple and low cost **setups** in coherent imaging :

(a) lensless microscopy, magnification $G = 1$

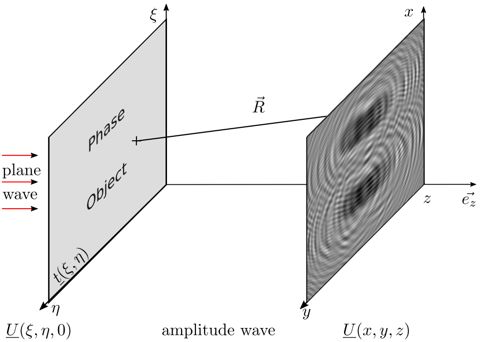
(b) lensless microscopy, magnification $G = \frac{z+z_s}{z} > 1$

(c) inline holographic microscopy magnification $G = 40, 60, 100$



Note : Colors are used in the figure for visualization reasons. The light is although monochromatic.

Inline Holography : Lensless microscopy **model**



$\underline{U}_z(x,y) = \underline{U}_0 \underset{x,y}{*} \underline{h}_z = \underline{A}.\left(\underline{t} \underset{x,y}{*} \underline{h}_z\right)$, the amplitude model is linear.

$I_z(x,y) = |\underline{U}_z(x,y)|^2$, the intensity model is not linear.

Inline Holography : Lensless microscopy **model**

Introducing the complex opacity function $\underline{\vartheta}$ such as $\underline{\vartheta} = 1 - \underline{t}$:

$\underline{U}_z(x,y) = \underline{U}_0 \underset{x,y}{*} \underline{h}_z = \underline{A}.\left((1 - \underline{\vartheta}) \underset{x,y}{*} \underline{h}_z \right)$

$I_z(x,y) \propto |1 - \underline{\vartheta} \underset{x,y}{*} \underline{h}_z|^2$

$= 1 + |\underline{\vartheta} \underset{x,y}{*} \underline{h}_z|^2$ incident wave intensity and diffracted wave intensity

$-\underline{\vartheta} \underset{x,y}{*} \underline{h}_z$ wave diffracted by the object

$-\underline{\vartheta}^* \underset{x,y}{*} \underline{h}_z^*$ wave diffracted by a virtual object (located at -z) of conjugate phase

Inline Holography : Lensless microscopy **model**

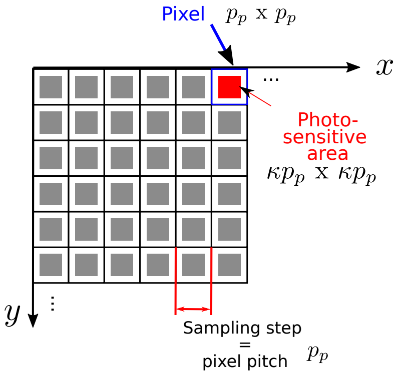
Before being sampled on the pixel grid, the signal is low-pass filtered on the pixel photosensitive area.

Model of the sensitive area effect :

$$I_z^\kappa(x,y) = \int_{x-\kappa p_p}^{x+\kappa p_p} \int_{y-\kappa p_p}^{y+\kappa p_p} I_z(x,y)\,dx\,dy$$

$$I_z^\kappa(x,y) = I_z \underset{x,y}{*} \Pi_{(\kappa.p_p,\kappa.p_p)}$$

κ^2 is called the fill factor (=pixel's light sensitive area to its total area).

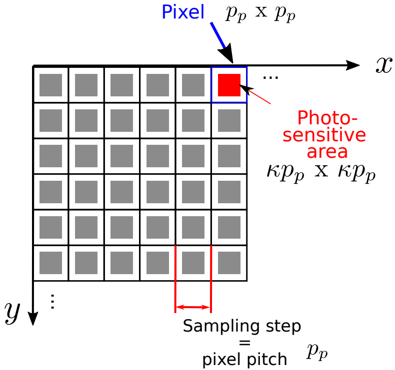


Inline Holography : Lensless microscopy **model**

After being low-pass filtered by the pixel photosensitive area, the intensity is sampled.

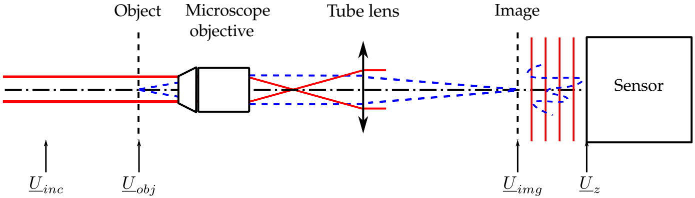
Sampling of the signal :
$I_z^{\kappa}(m,n) = I_z^{\kappa}(x,y).Ш_{p_p,p_p}(x,y)$
Filtering + Sampling

$I_z^{\kappa}(m,n) = \left(I_z \underset{x,y}{*} \Pi_{(\kappa.p_p,\kappa.p_p)} \right).Ш_{p_p,p_p}(x,y)$



$$Ш_{p_p,p_p}(x,y) = \begin{cases} 1 & \text{if } x = mp_p \text{ and } y = np_p \\ 0 & \text{else} \end{cases}$$

Inline Holographic Microscopy **setup**



$\underline{U}_z(x,y) = \underline{U}_{img} \underset{x,y}{*} \underline{h}_z$

Taking into account only the magnification of the image :

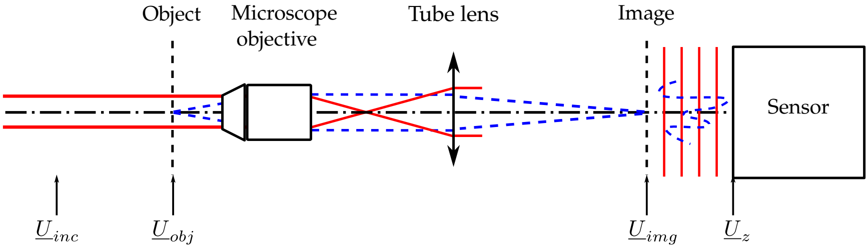
$\underline{U}_{img}(x,y) = \frac{1}{G}\underline{U}_{obj}(x/G, y/G)$ with $\underline{U}_{obj}(x,y) = \underline{U}_{inc}.\underline{t}_{obj}(x,y)$

Taking into account the numerical aperture of the objective :

$\underline{U}^{f}_{img}(x,y) = \underline{U}_{img} \underset{x,y}{*} \text{PSF}$

$$\underline{U}_z(x,y) = \underbrace{\frac{1}{G}\underline{U}_{inc}.\underline{t}_{obj}(\xi/G, \eta/G)}_{\text{Magnified object}} \underset{x,y}{*} \underbrace{\left(\text{PSF} \underset{\xi,\eta}{*} \underline{h}_z\right)}_{\text{New PSF}}$$

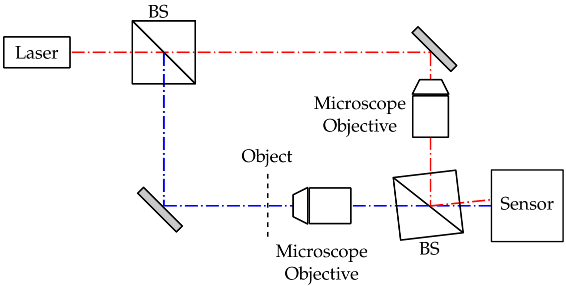
Inline Holographic Microscopy **model**

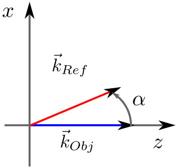


$$I_z(x,y) = |\underline{U}_z(x,y)|^2 = |\underbrace{\frac{1}{G}\underline{U}_{inc}.\underline{t}_{obj}(\xi/G,\eta/G)}_{\text{Magnified object}} \underset{x,y}{*} \underbrace{\left(\text{PSF} \underset{\xi,\eta}{*} \underline{h}_z\right)}_{\text{New PSF}}|^2$$

In the case of z=0, and a point source object (modeled by a Dirac function) : $I_z(x,y) = |\text{PSF}(x,y)|^2$

Off-axis Holographic microscopy **setup**





The interferences between the object wave (in blue) and the reference wave (in red) are modeled by :

$I = |\underline{u}_{Obj} + \underline{u}_{Ref}|^2 = |\underline{u}_{Obj}|^2 + |\underline{u}_{Ref}|^2 + \underline{u}_{Obj}.\underline{u}_{Ref}{}^* + \underline{u}_{Ref}.\underline{u}_{Obj}{}^*$

$\underline{u}_{Ref}(x, y) = e^{-i(k_x^{Ref}x + k_y^{Ref}y)}$

$$\underline{u}_{Obj}(x, y) = \underline{u}_z(x, y) = \underline{u}_{inc}.\underline{t}_{img} \underset{x,y}{*} \left(\mathrm{PSF} \underset{\xi,\eta}{*} \underline{h}_z \right)$$

Off-axis Holographic microscopy **model**

To simplify and shorten the mathematical notations let us assume $k_x^{Ref}x + k_y^{Ref}y = k_x x$

$$I(x,y) =$$

$$\underbrace{|\underline{U}_{Obj}(x,y)|^2 + |e^{-ik_x x}|^2}_{(\mathbf{0})} + \underbrace{\underline{U}_{Obj}(x,y).e^{ik_x x}}_{(+\mathbf{1})} + \underbrace{\underline{U}_{Obj}{}^*(x,y).e^{-ik_x x}}_{(-\mathbf{1})}$$

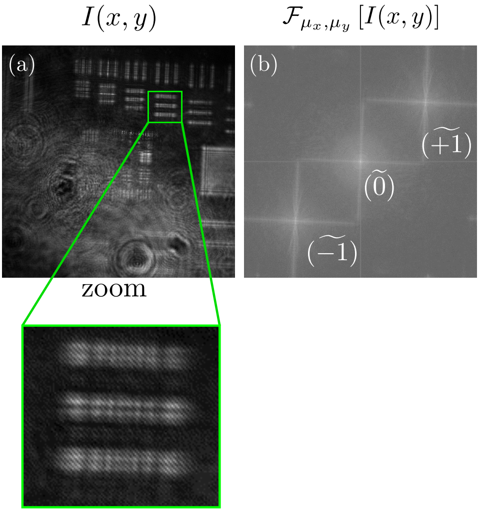
$$\mathcal{F}_{\mu_x,\mu_y}\left[I(x,y)\right] =$$

$$\underbrace{|\widetilde{\underline{U}_{Obj}}(\mu_x,\mu_y)|^2 + \mathcal{F}(1)}_{(\widetilde{\mathbf{0}})} + \underbrace{\widetilde{\underline{U}_{Obj}}{}^*(\mu_x,\mu_y) * \delta(\mu_x + k_x x, \mu_y)}_{(\widetilde{+\mathbf{1}})}$$

$$+ \underbrace{\widetilde{\underline{U}_{Obj}}(\mu_x,\mu_y) * \delta(\mu_x - k_x x, \mu_y)}_{(\widetilde{-\mathbf{1}})}$$

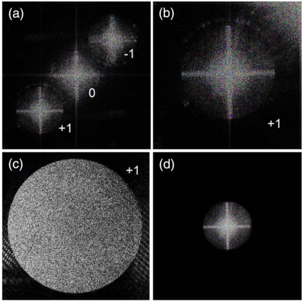
Off-axis Holographic microscopy **model** illustration 1

Images examples in the case $k_x^{Ref} = k_y^{Ref}$:



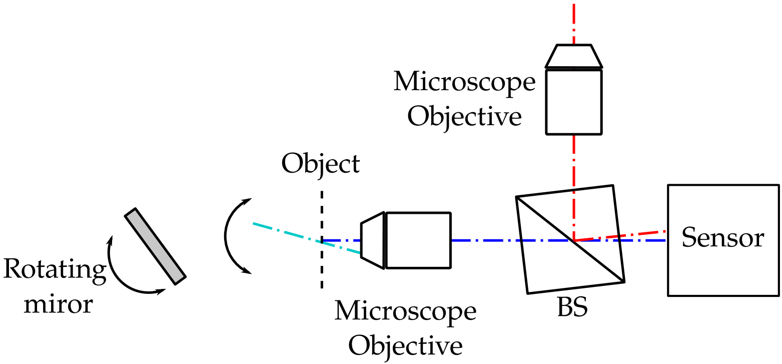
Off-axis Holographic microscopy **model** illustration 2

$$\mathcal{F}_{\mu_x,\mu_y}\left[I(x,y)\right]$$



* Verrier, Nicolas, et al. "Holographic microscopy reconstruction in both object and image half-spaces with an undistorted three-dimensional grid." Applied optics,2015.

Tomographic Diffractive microscopy **setup**



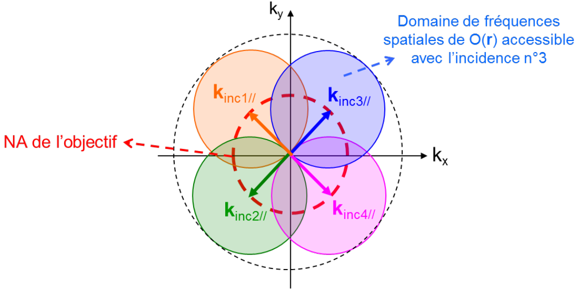
Tomographic Diffractive microscopy **model**

$$\underline{U}_{Obj}(x,y) = \underline{U}_z(x,y) = \left(\underline{U}_{inc}(\xi,\eta).\underline{t}_{img}(\xi,\eta)\right) \underset{x,y}{*} \left(\mathrm{PSF} \underset{\xi,\eta}{*} \underline{h}_z\right)$$

$$\underline{U}_{Obj}(x,y,0) = \left(\underline{U}^0_{inc} e^{jk_{inc}\xi}.\underline{t}_{img}(\xi,\eta)\right) \underset{x,y}{*} \mathrm{PSF}(\xi,\eta)$$

$$\widetilde{\underline{U}_{Obj}}(\mu_x,\mu_y) = \left(\underline{U}^0_{inc}\delta_{(k_{inc}/2\pi,0)} \underset{\mu_x,\mu_y}{*} \widetilde{\underline{t}_{img}}\right).\widetilde{\mathrm{PSF}}(\mu_x,\mu_y)$$

Shift of the transmittance spectrum in Fourier domain →Super-resolution (numerical aperture is multiplied by 2).

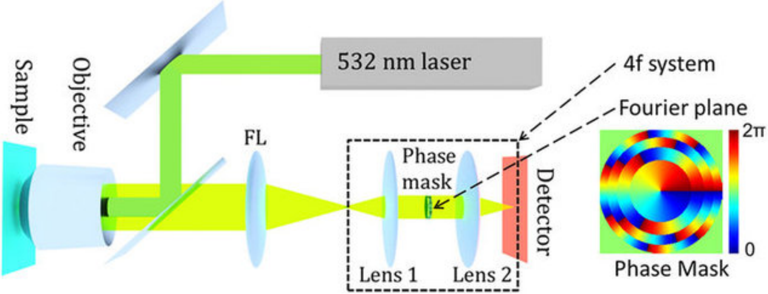


+ 3D information...

Image formation : Phase masks and aberrations

Phase masks (setup)

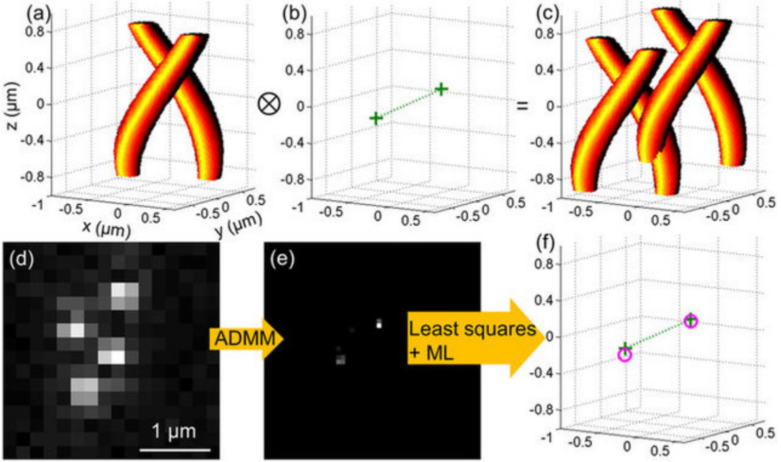
Using a 4f setup, and a phase mask, it is possible to co-design a 3D PSF –> give accurate positioning of objects in 3D.



Ref : Shuang, Bo, et al. "Generalized recovery algorithm for 3D super-resolution microscopy using rotating point spread functions." Scientific reports 6.1 (2016) : 1-9.

Phase masks : 3D PSF

Example of 3D PSF



Ref : Shuang, Bo, et al. "Generalized recovery algorithm for 3D super-resolution microscopy using rotating point spread functions." Scientific reports 6.1 (2016) : 1-9.

Phase masks : model

The phase mask acts in Fourier Domain of the image. Taking into account only the magnification G of the image :
$\underline{U}_{img}^{G}(x, y) = \underline{U}_{obj}(x/G, y/G)/G$
Taking into account the numerical aperture (NA) of the objective :
$\underline{U}_{img}^{PSF}(x, y) = \underline{U}_{img}^{G} \underset{x,y}{*} PSF_{NA}$
$\underline{U}_{img}^{PSF}(x, y) = \mathcal{F}_{x,y}^{-1}\left(\underline{\tilde{U}}_{img}(\mu_x, \mu_y).P\tilde{S}F_{NA}(\mu_x, \mu_y)\right)$

Taking into account the mask in Fourier space :
$\underline{U}_{img}^{Mask}(x, y) = \underline{U}_{img}^{PSF} \underset{x,y}{*} \underline{PSF}_{Mask}$
$\underline{U}_{img}^{Mask}(x, y) = \mathcal{F}_{x,y}^{-1}\left(\underline{\tilde{U}}_{img}^{PSF}(\mu_x, \mu_y).P\tilde{S}F_{Mask}(\mu_x, \mu_y)\right)$
$\underline{U}_{img}^{Mask} = \underline{U}_{img}^{G} * \underline{PSF}_{NA} * PSF_{Mask}$

Phase masks : codesign

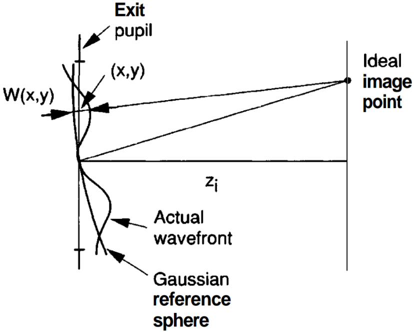
Phase mask can be used to record 3D information in the image, to improve axial localization/resolution [2] of an image.
See Pauline Trouvé's lecture.

2. Verrier, Nicolas, et al. "Co-design of an in-line holographic microscope with enhanced axial resolution : selective filtering digital holography." JOSA A, 2016

Model of aberration

Vocabulary : a diffraction limited system assume that there is no aberration.

Aberrations distort the exit-pupil wavefront. It is no more a truncated spherical wavefront.
It can lead, e.g. to defocus error (spherical aberration).
The aberrations can be modeled by a phase mask located in the exit-pupil plane.



Model of aberration

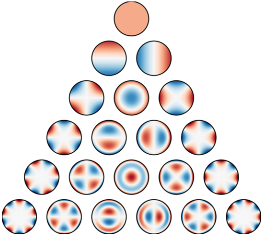
Distortion of the exit-pupil wavefront can be modeled by a "virtual" shifting phase plate in the aperture deforming the wave front that leaves the pupill. Let's consider the path-length error $W^{err}(x, y)$ at point (x, y). The complex transmittance of the imaginary shifting plate, called the generalized pupil function is given by :

$\underline{P^{err}}(x, y) = P(x, y)e^{jkW^{err}(x,y)}$

The coherent PSF is given by :

$\underline{PSF} = \mathcal{F}(\underline{P^{err}})$

In Fourier domain the aberrations can be modeled by Zernike polynoms.



Incoherent imaging

Incoherent PSF

Incoherent PSF and Optical Transfer Function

When using incoherent illumination, the various impulse responses in the image plane vary in uncorrelated fashions. They must therefore be added on intensity basis.

In space domain :

$$I_{img}(x, y) = I_{obj} \underset{x,y}{*} |\underline{PSF^{coh}}|^2$$

In Fourier domain :

$$\widetilde{I_{img}}(\mu_x, \mu_y) = \widetilde{I_{obj}}(\mu_x, \mu_y). \underbrace{\left(\underline{\widetilde{PSF^{coh}}} \underset{(\mu_x,\mu_y)}{*} \underline{\widetilde{PSF^{coh}}}^{*}\right)}_{\text{Transfer Function}}$$

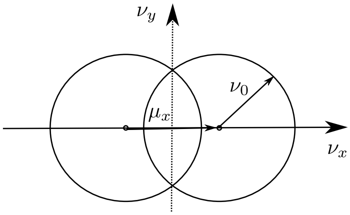
The normalized incoherent Transfert function is called the Optical Transfer function OTF)

Incoherent PSF and Optical Transfer Function

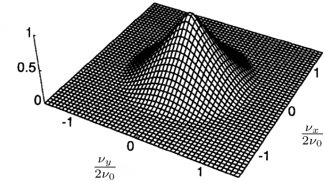
For a diffraction limited incoherent system :

$$\widetilde{\mathrm{PSF}}^{coh}(\mu_x, \mu_y) = P(\lambda z_i \mu_x, \lambda z_i \mu_y)$$

$$\widetilde{\mathrm{PSF}}^{incoh}(\mu_x, \mu_y) = P(\lambda z_i \nu_x, \lambda z_i \nu_y) \underset{(\mu_x, \mu_y)}{*} P(\lambda z_i \mu_x, \lambda z_i \mu_y)$$



$\widetilde{PSF}^{inc}$ normalized =OTF



$\widetilde{PSF}^{inc}$ normalized= OTF