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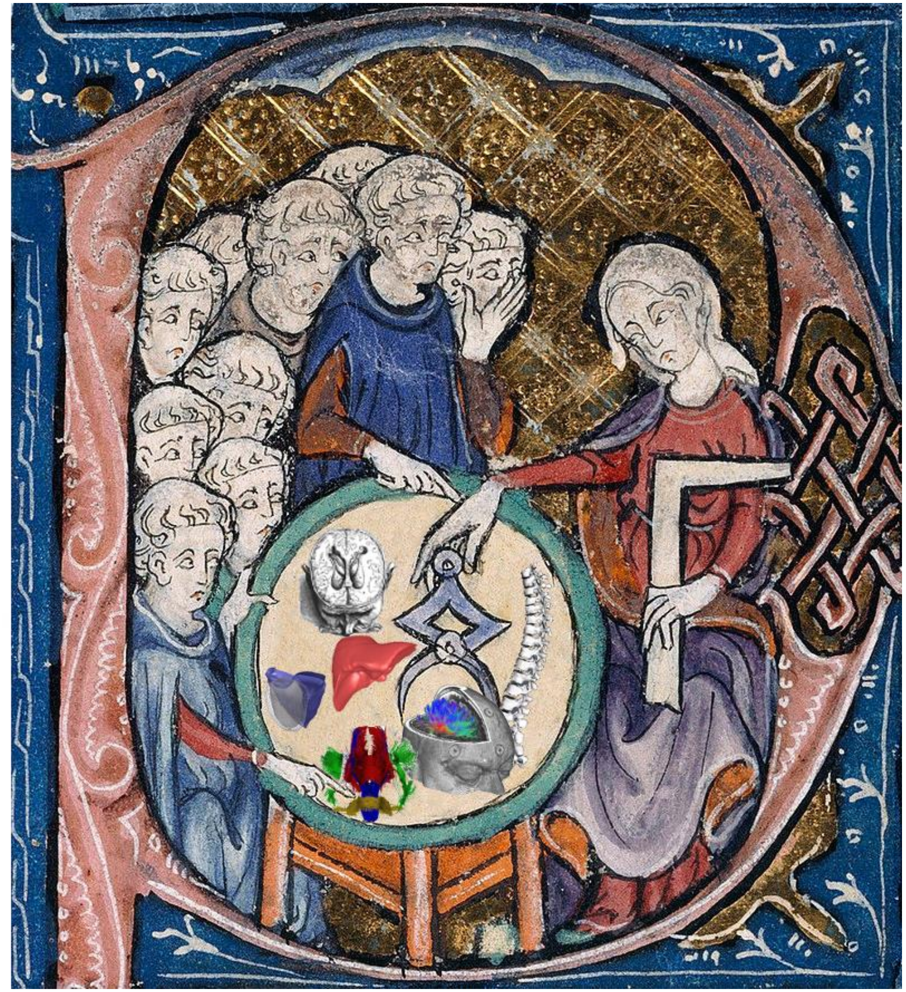
http://www-sop.inria.fr/asclepios/cours/Peyresq_2019/

Geometric Statistics

Mathematical foundations and applications in computational anatomy



Freely adapted from "Women teaching geometry", in Adelard of Bath translation of Euclid's elements, 1310.

3/ Metric and Affine Geometric Settings for Lie Groups

Ecole d'été de Peyresq, Jul 1-5 2019

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Manifold-Valued Image Processing

Metric and Affine Geometric Settings for Lie Groups

Parallel transport to analyze Longitudinal deformations

Advances Statistics: CLT & PCA

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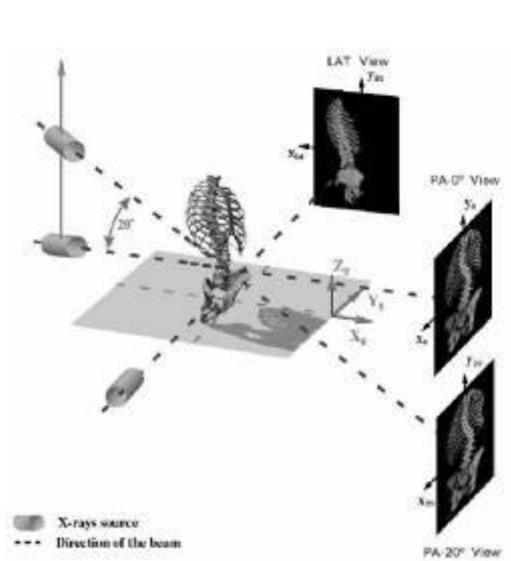
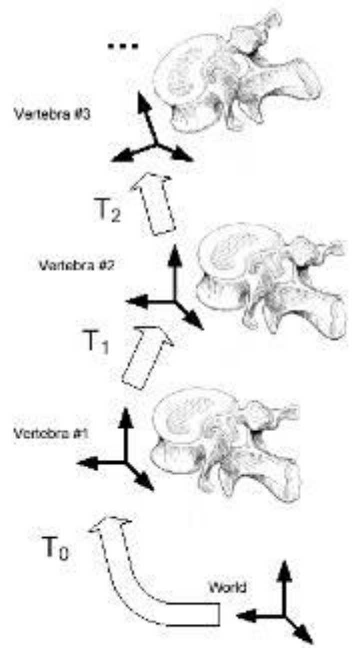
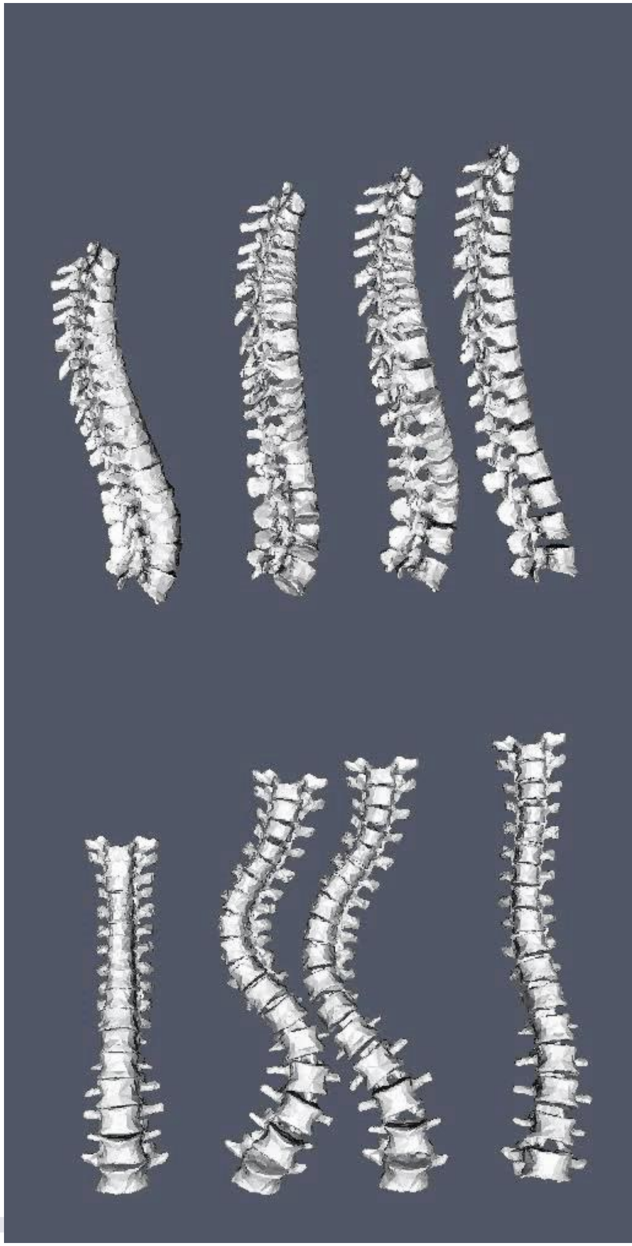
Statistical Analysis of the Scoliotic Spine







Data

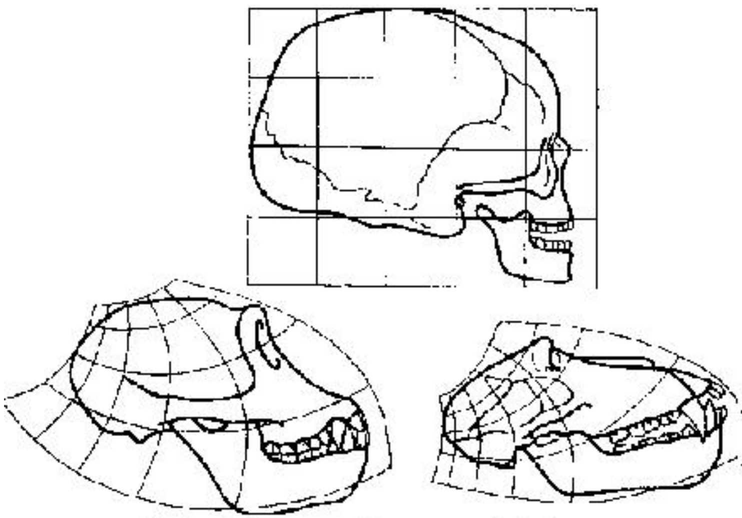
- 307 Scoliotic patients from the Montreal's St-Justine Hosp
- 3D Geometry from multi-planar X-rays
- Articulated model:17 relative pose of successive vertebras

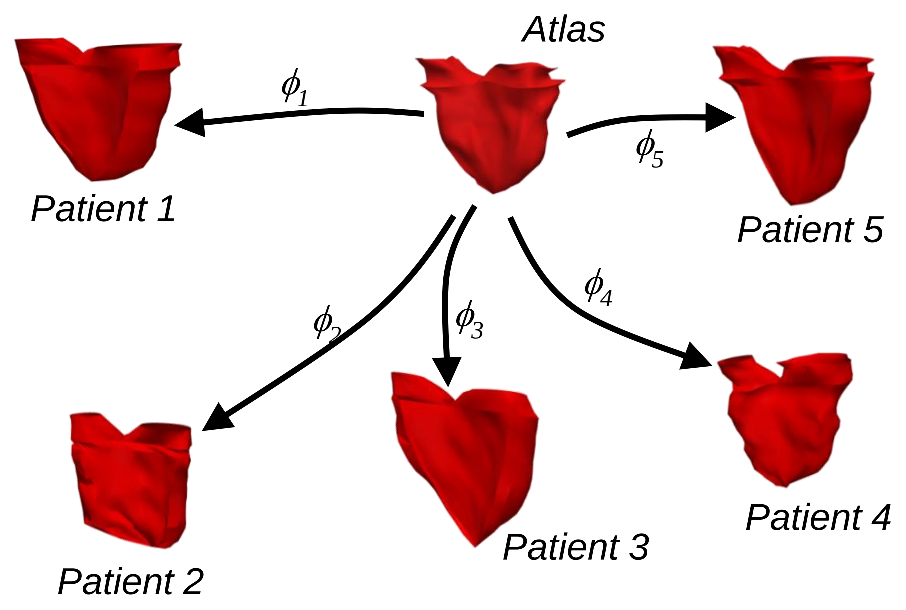
Statistics

- Main translation variability is axial (growth?)
- Main rot. var. around anterior-posterior axis
- **4 first variation modes related to King's classes**

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]

Morphometry through Deformations





Measure of deformation [D'Arcy Thompson 1917, Grenander & Miller]

- Observation = "random" deformation of a reference template
- Deterministic template = anatomical invariants [Atlas ~ mean]
- Random deformations = geometrical variability [Covariance matrix]

Natural Riemannian Metrics on Transformations

Transformation are Lie groups: Smooth manifold G compatible with group structure

- Composition $g \circ h$ and inversion g^{-1} are smooth
- Left and Right translation $L_g(f) = g \circ f \quad R_g(f) = f \circ g$
- Conjugation $Conj_g(f) = g \circ f \circ g^{-1}$

Natural Riemannian metric choices

- Chose a metric at Id: $\langle x,y \rangle_{Id}$
- Propagate at each point g using left (or right) translation
  $\langle x,y \rangle_g = \langle DL_{g^{(-1)}} .x \,,\, DL_{g^{(-1)}} .y \rangle_{Id}$

Implementation

- Practical computations using left (or right) translations

$$\mathrm{Exp}_f(x) = f \circ \mathrm{Exp}_{Id}(DL_{f^{(-1)}}.x) \qquad \overrightarrow{fg} = \mathrm{Log}_f(g) = DL_f.\mathrm{Log}_{Id}(f^{(-1)} \circ g)$$

General Non-Compact and Non-Commutative case

No Bi-invariant Mean for 2D Rigid Body Transformations

- Metric at Identity: $dist(Id, (\theta; t_1; t_2))^2 = \theta^2 + t_1^2 + t_2^2$

- $T_1 = \left(\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$ $\quad T_2 = \left(0; \sqrt{2}; 0\right)$ $\quad T_3 = \left(-\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}\right)$

- Left-invariant Fréchet mean: $(0; 0; 0)$
- Right-invariant Fréchet mean: $\left(0; \frac{\sqrt{2}}{3}; 0\right) \simeq (0; 0.4714; 0)$

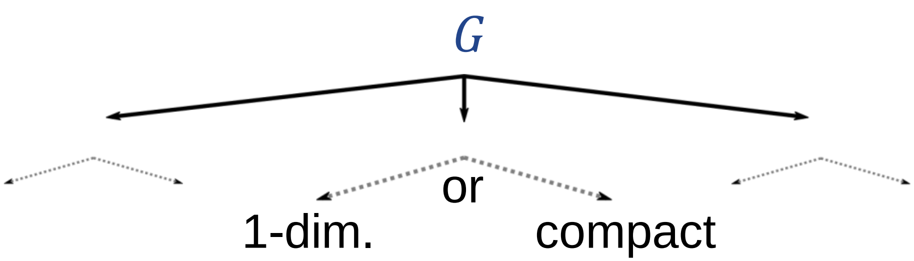
Questions for this talk:

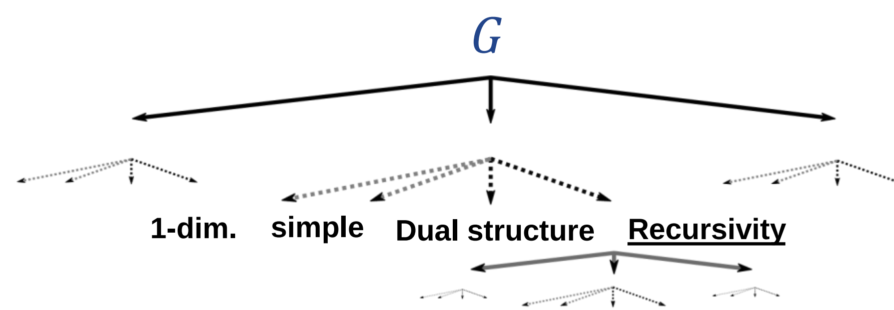
- **Can we design a mean compatible with the group operations?**
- **Is there a more convenient structure for statistics on Lie groups?**

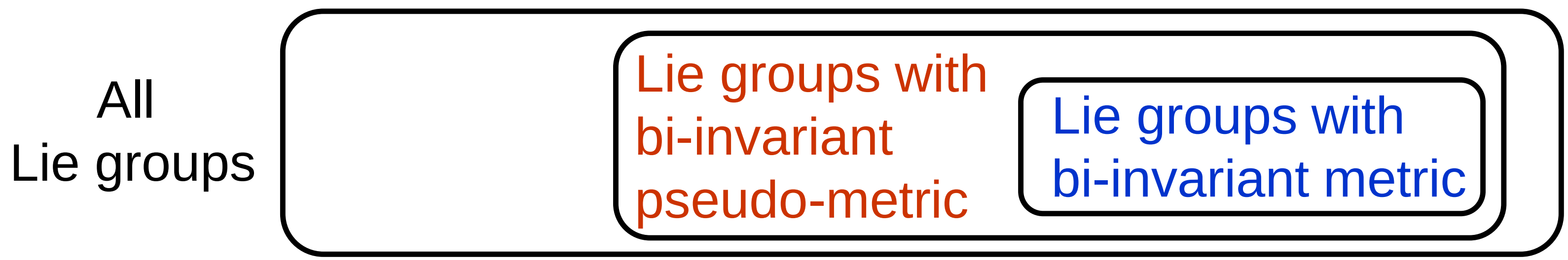
Existence of bi-invariant (pseudo) metrics

[Cartan 50's]:
Bi-invariant metric on G

[Medina, Revoy 80's]:
Bi-invariant pseudo-metric on G







[Miolane, Pennec, Computing Bi-Invariant Pseudo-Metrics on Lie Groups for Consistent Statistics. Entropy, 17(4):1850-1881, April 2015.]

- Algorithm: decompose the Lie algebra and find a bi-inv. pseudo-metric
- Test on rigid transformations SE(n): bi-inv. ps-metric for n=1 or 3 only

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Basics of Lie groups

Flow of a left invariant vector field $\tilde{X} = DL.x$ from identity

- $\gamma_x(t)$ exists for all time
- One parameter subgroup: $\gamma_x(s+t) = \gamma_x(s).\,\gamma_x(t)$

Lie group exponential

- Definition: $x \in \mathfrak{g} \rightarrow Exp(x) = \gamma_x(1) \epsilon G$
- Diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G (not true in general for inf. dim)

3 curves parameterized by the same tangent vector

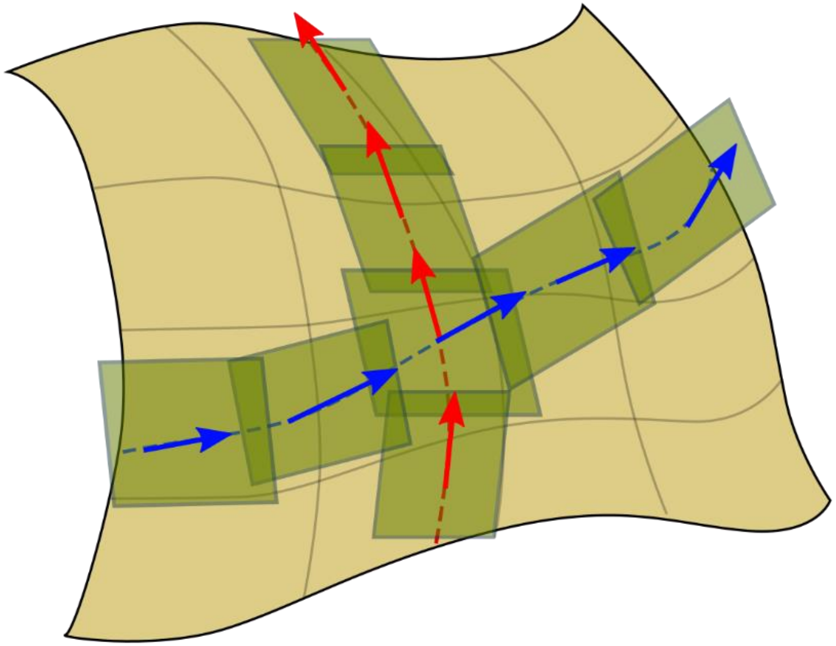
- Left / Right-invariant geodesics, one-parameter subgroups

Question: Can one-parameter subgroups be geodesics?

Affine connection spaces: Drop the metric, use connection to define geodesics

Affine Connection (infinitesimal parallel transport)

- Acceleration = derivative of the tangent vector along a curve
- Projection of a tangent space on a neighboring tangent space



Geodesics = straight lines

- Null acceleration: $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
- 2nd order differential equation: Normal coordinate system
- Local exp and log maps

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

Canonical Affine Connections on Lie Groups

A unique Cartan-Schouten connection

- Bi-invariant and symmetric (no torsion)
- Geodesics through Id are one-parameter subgroups (group exponential)
  - Matrices : M(t) = A exp(t.V)
  - Diffeos : **translations of Stationary Velocity Fields (SVFs)**

Levi-Civita connection of a bi-invariant metric (if it exists)

- Continues to exists in the absence of such a metric (e.g. for rigid or affine transformations)

Symmetric space with central symmetry $\boldsymbol{S_{\psi}(\phi) = \psi\phi^{-1}\psi}$

- Matrix geodesic symmetry: $S_A\big(M(t)\big) = A\exp(-tV)A^{-1}A = M(-t)$

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

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Mean value on an affine connection space

Fréchet / Karcher means not usable (no distance) but:

$$\mathrm{E}[\mathbf{x}] = \underset{y \in \mathrm{M}}{\operatorname{argmin}} \left(\mathrm{E}\left[\operatorname{dist}(y, \mathbf{x})^2 \right] \right) \quad \Rightarrow \quad \mathrm{E}\left[\overrightarrow{\bar{\mathrm{x}}\mathbf{x}} \right] = \int_{\mathrm{M}} \overrightarrow{\bar{\mathrm{x}}\mathbf{x}} . p_{\mathbf{x}}(z).d\mathrm{M}(z) = 0 \quad [P(C) = 0]$$

Exponential barycenters

- [Emery & Mokobodzki 91, Corcuera & Kendall 99]

$$\int Log_x(y)\, \mu(dy) = 0 \quad \text{or} \quad \sum_i Log_x(y_i) = 0$$

- Existence? Uniqueness?
- OK for convex affine manifolds with semi-local convex geometry [Arnaudon & Li, Ann. Prob. 33-4, 2005]
  - Use a separating function (convex function separating points) instead of a distance
- Algorithm to compute the mean: fixed point iteration (stability?)

Bi-invariant Mean on Lie Groups

Exponential barycenter of the symmetric Cartan connection

- Locus of points where $\sum Log(m^{-1}.g_i) = 0$ (whenever defined)
- Iterative algorithm: $m_{t+1} = m_t \circ Exp\left(\frac{1}{n}\sum Log(m_t^{-1}.g_i)\right)$
- First step corresponds to the Log-Euclidean mean
- Corresponds to the first definition of bi-invariant mean of [V. Arsigny, X. Pennec, and N. Ayache. Research Report RR-5885, INRIA, April 2006.]

Mean is stable by left / right composition and inversion

- If m is a mean of $\{g_i\}$ and h is any group element, then
  - $h \circ m$ is a mean of $\{h \circ g_i\}$,
  - $m \circ h$ is a mean of the points $\{g_i \circ h\}$
  - and $m^{(-1)}$ is a mean of $\left\{g_i^{(-1)}\right\}$

[XP and V. Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Matrix Information Geometry. 2012]

Bi-invariant Mean on Lie Groups

Fine existence

- If the data points belong to a sufficiently small normal convex neighborhood of some point, then there exists a unique solution in this NCN.
- Moreover, the iterated point strategy converges at least at a linear rate towards this unique solution, provided the initialization is close enough.

- Proof: using an auxiliary metric, the iteration is a contraction.

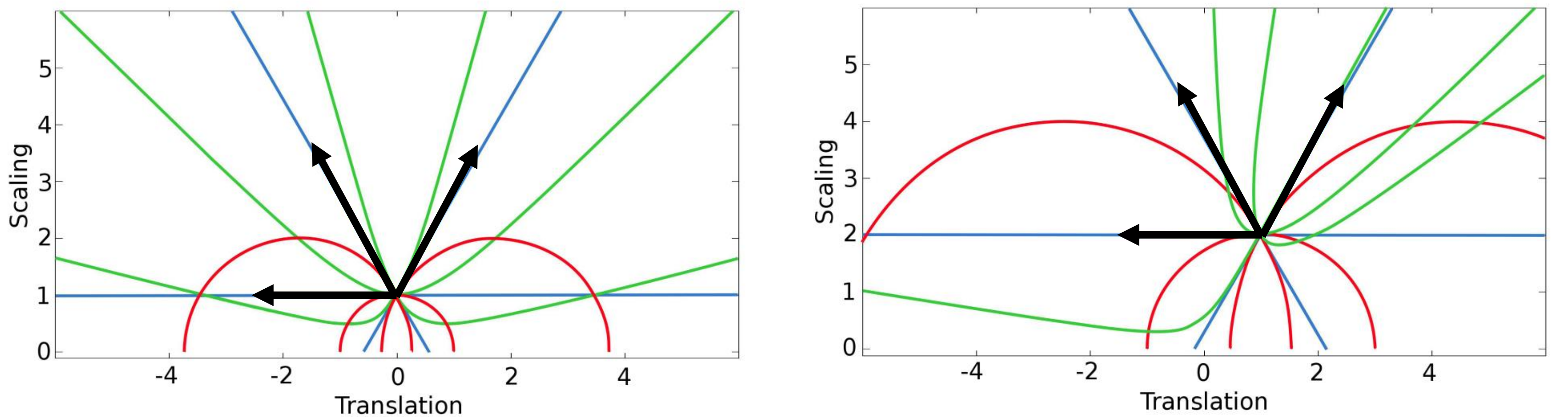
Closed-form for 2 points

- $m(t) = x \circ Exp\,(t.Log(x^{(-1)} \circ y))$

Special Matrix Groups

Scaling and translations ST(n)

- No bi-invariant metric
- Group geodesics defined globally, all points are reachable
- Existence and uniqueness of bi-invariant mean (closed form)



Group / left-invariant / right-invariant geodesics

Special matrix groups

Heisenberg Group (resp. Scaled Upper Unitriangular Matrix Group)

- No bi-invariant metric
- Group geodesics defined globally, all points are reachable
- Existence and uniqueness of bi-invariant mean (closed form resp. solvable)

Rigid-body transformations

- Logarithm well defined iff log of rotation part is well defined, i.e. if the 2D rotation have angles $|\theta_i| < \pi$
- Existence and uniqueness with same criterion as for rotation parts (same as Riemannian)

SU(n) and GL(n):

- log does not always exist (need 2 exp to cover)

Example mean of 2D rigid-body transformation

$$T_1 = \left(\frac{\pi}{4};\ -\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right) \qquad T_2 = (0;\ \sqrt{2}; 0\ ) \qquad T_3 = \left(-\frac{\pi}{4};\ -\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}\right)$$

- Metric at Identity: $dist(Id, (\theta; t_1; t_2))^2 = \theta^2 +\ t_1^2 + t_2^2$
- Left-invariant Fréchet mean: $(0; 0; 0\ )$
- Log-Euclidean mean: $\left(0; \frac{\sqrt{2}-\pi/4}{3}; 0\right) \simeq (0; 0.2096; 0)$
- Bi-invariant mean: $\left(0; \frac{\sqrt{2}-\pi/4}{1+\pi/4(\sqrt{2}+1)}; 0\right) \simeq (0; 0.2171; 0)$
- Right-invariant Fréchet mean: $\left(0; \frac{\sqrt{2}}{3}; 0\right) \simeq (0; 0.4714; 0)$

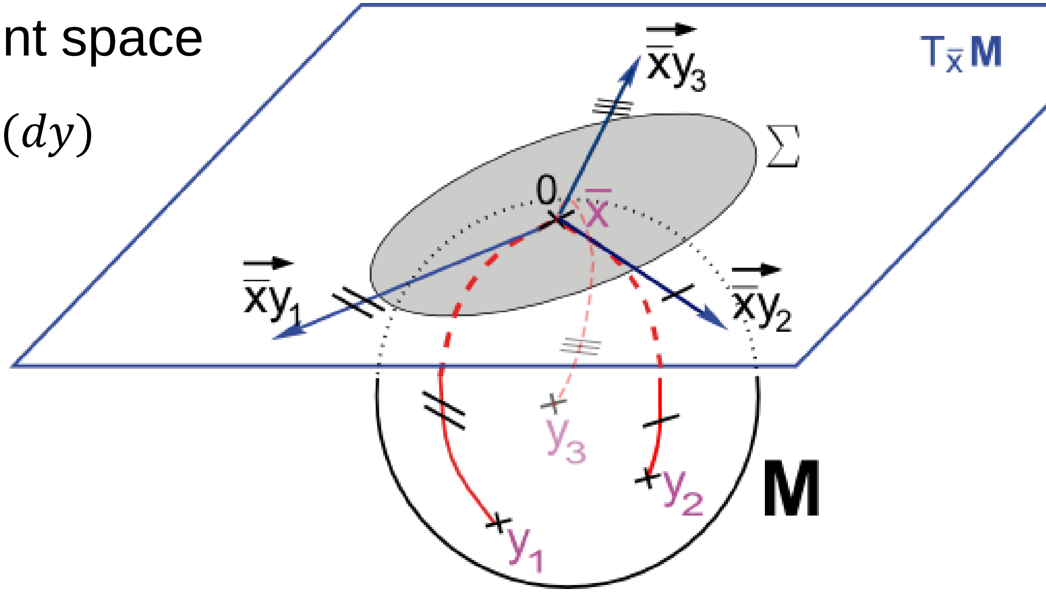
Generalization of the Statistical Framework

Covariance matrix & higher order moments

- Defined as tensors in tangent space

$$\Sigma = \int Log_x(y) \otimes Log_x(y)\, \mu(dy)$$

- Matrix expression changes according to the basis



Other statistical tools

- Mahalanobis distance well defined and bi-invariant

$$\mu_{(m,\Sigma)}(g) = \int [Log_m(g)]^i \Sigma_{ij}^{(-1)} [Log_m(g)]^j \mu(dy)$$

- ~~Tangent Principal Component Analysis (t-PCA)~~
- Principal Geodesic Analysis (PGA), provided a data likelihood
- Independent Component Analysis (ICA)

Cartan Connections vs Riemannian

What is similar

- Standard differentiable geometric structure [curved space without torsion]
- Normal coordinate system with Exp_x et Log_x [finite dimension]

Limitations of the affine framework

- No metric (but no choice of metric to justify)
- The exponential does always not cover the full group
  - Pathological examples close to identity in finite dimension
  - In practice, similar limitations for the discrete Riemannian framework
- Global existence and uniqueness of bi-invariant mean?
  Use a bi-invariant pseudo-Riemannian metric? [Miolane MaxEnt 2014]

What we gain

- A globally invariant structure invariant by composition & inversion
- Simple geodesics, efficient computations (stationarity, group exponential)
- The simplest linearization of transformations for statistics?

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Riemannian Metrics on diffeomorphisms

Space of deformations

- Transformation y=ϕ(x)
- Curves in transformation spaces: ϕ(x,t)
- Tangent vector = speed vector field

$$v_t(x) = \frac{d\phi(x,t)}{dt}$$

Right invariant metric

$$\left\|v_t\right\|_{\phi_t} = \left\|v_t \circ \phi_t^{-1}\right\|_{Id}$$

- Eulerian scheme
- Sobolev Norm H_k or H_∞ (RKHS) in LDDMM → diffeomorphisms [Miller, Trouve, Younes, Holm, Dupuis, Beg… 1998 – 2009]

Geodesics determined by optimization of a time-varying vector field

- Distance

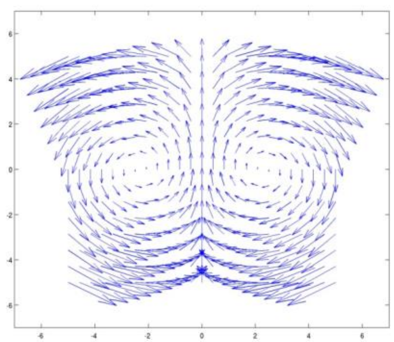
$$d^2(\phi_0, \phi_1) = \arg\min_{v_t}\left(\int_0^1 \left\|v_t\right\|_{\phi_t}^2 dt\right)$$

- Geodesics characterized by initial velocity / momentum
- Optimization for images is quite tricky (and lenghty)

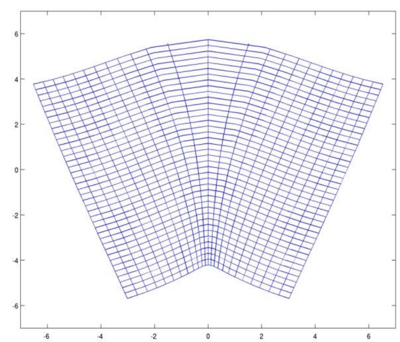
The SVF framework for Diffeomorphisms

Idea: **[Arsigny MICCAI 2006, Bossa MICCAI 2007, Ashburner Neuroimage 2007]**

- Exponential of a smooth vector field is a diffeomorphism
- Parameterize deformation by ~~time-varying~~ Stationary Velocity Fields



Stationary velocity field



Diffeomorphism

Direct generalization of numerical matrix algorithms

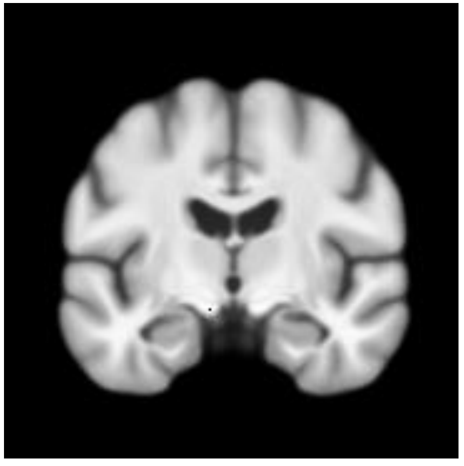
- Computing the deformation: **Scaling and squaring** **[Arsigny MICCAI 2006]**
  recursive use of exp($\boldsymbol{v}$)=exp($\boldsymbol{v}$/2) o exp($\boldsymbol{v}$/2)
- Computing the Jacobian :Dexp(v) = Dexp(v/2) o exp(v/2) . Dexp(v/2)
- Updating the deformation parameters: **BCH formula** **[Bossa MICCAI 2007]**

  $\exp(\boldsymbol{v}) \circ \exp(\varepsilon \boldsymbol{u}) = \exp(\ \boldsymbol{v} + \varepsilon \boldsymbol{u} + [\boldsymbol{v},\varepsilon \boldsymbol{u}]/2 + [\boldsymbol{v},[\boldsymbol{v},\varepsilon \boldsymbol{u}]]/12 + \dots\ )$

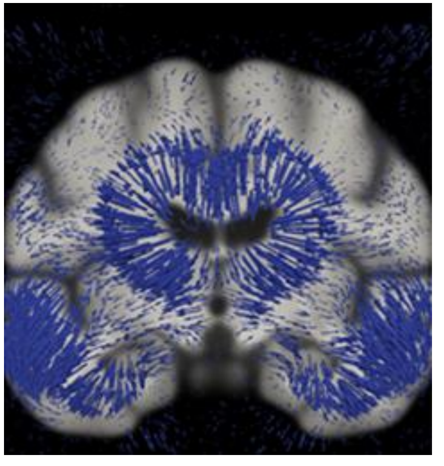
  - Lie bracket $[\boldsymbol{v},\boldsymbol{u}](p) = \mathrm{Jac}(\boldsymbol{v})(p).\boldsymbol{u}(p) - \mathrm{Jac}(\boldsymbol{u})(p).\boldsymbol{v}(p)$

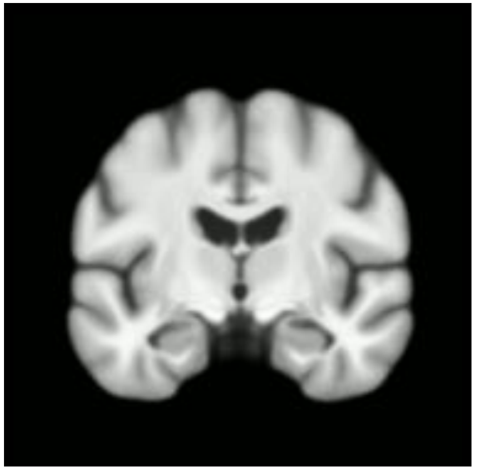
Measuring Temporal Evolution with deformations

Optimize LCC with deformation parameterized by SVF



$$\varphi_t(x) = exp(t.v(x))$$





https://team.inria.fr/asclepios/software/lcclogdemons/

[Lorenzi, Ayache, Frisoni, Pennec, Neuroimage 81, 1 (2013) 470-483]

The Stationnary Velocity Fields (SVF) framework for diffeomorphisms

- SVF framework for diffeomorphisms is algorithmically simple
- Compatible with “inverse-consistency”
- Vector statistics directly generalized to diffeomorphisms.

Registration algorithms using log-demons:

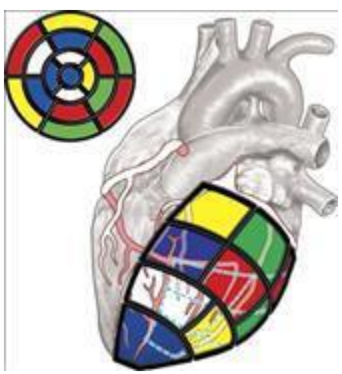
- Log-demons: Open-source ITK implementation (Vercauteren MICCAI 2008)
  http://hdl.handle.net/10380/3060
  [MICCAI Young Scientist Impact award 2013]
- Tensor (DTI) Log-demons (Sweet WBIR 2010):
  https://gforge.inria.fr/projects/ttk
- LCC log-demons for AD (Lorenzi, Neuroimage. 2013)
  https://team.inria.fr/asclepios/software/lcclogdemons/
- 3D myocardium strain / incompressible deformations (Mansi MICCAI’10)
- Hierarchichal multiscale polyaffine log-demons (Seiler, Media 2012)
  http://www.stanford.edu/~cseiler/software.html
  [MICCAI 2011 Young Scientist award]

A powerful framework for statistics

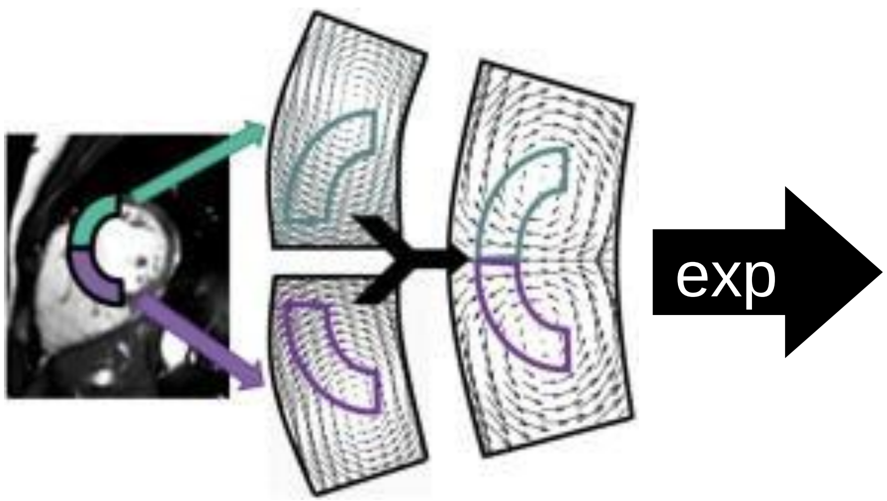
Parametric diffeomorphisms [Arsigny et al., MICCAI 06, JMIV 09]

- One affine transformation per region (polyaffines transformations)
- Cardiac motion tracking for **each subject** [McLeod, Miccai 2013]

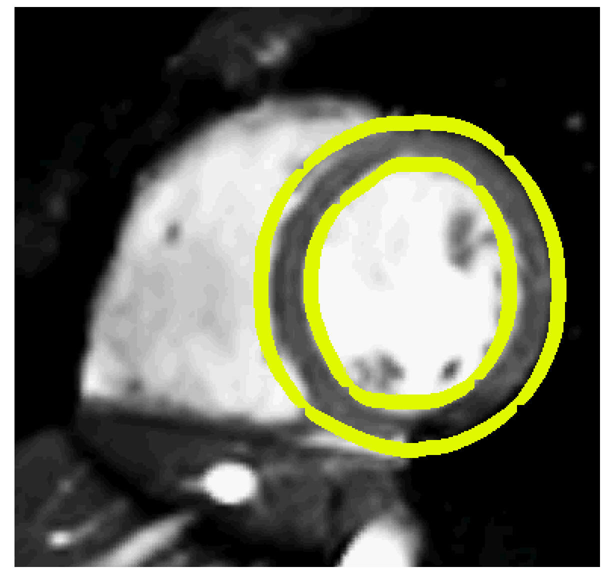
Log demons projected but with 204 parameters instead of a few millions



AHA regions



Stationary velocity fields



Diffeomorphism

A powerful framework for statistics

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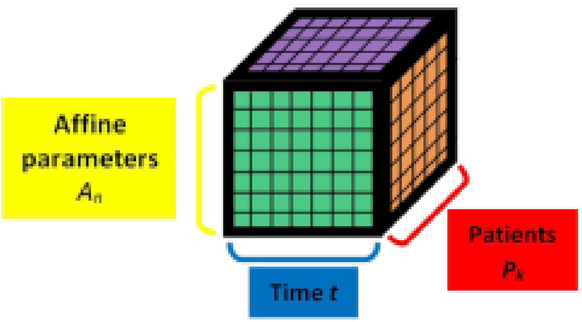
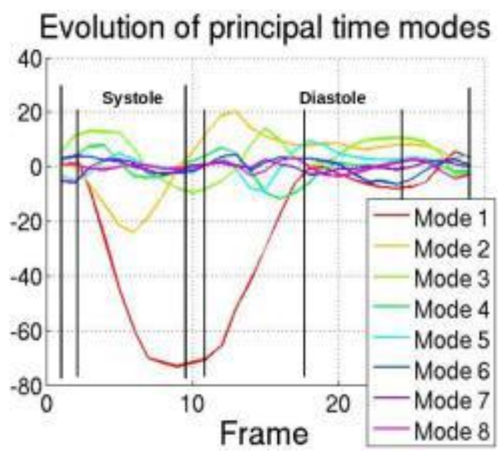
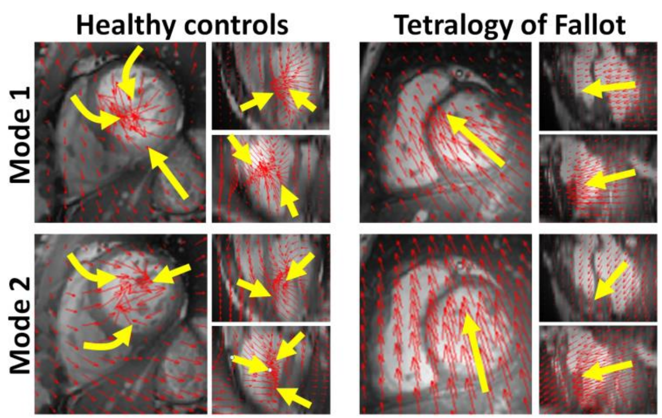
Log demons projected but with 204 parameters instead of a few millions

- **Group analysis** using tensor reduction : reduced model

8 temporal modes x 3 spatial modes = 24 parameters (instead of 204)



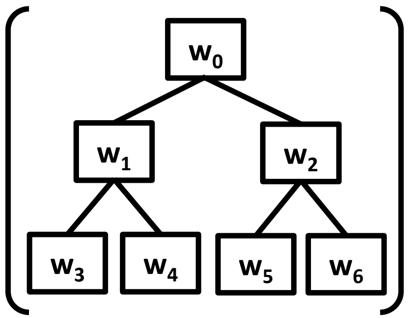


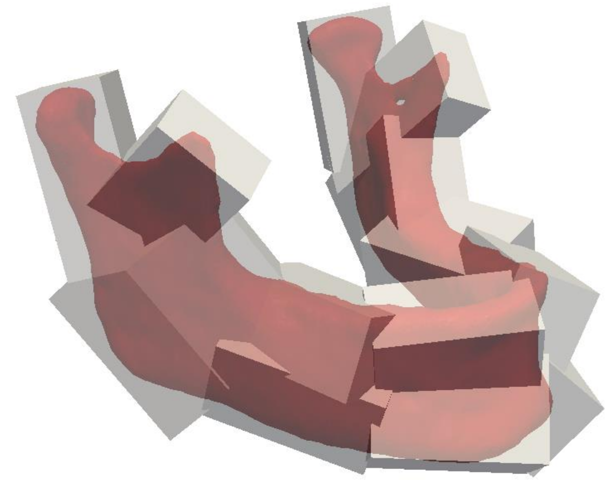


Hierarchical Deformation model

Population level:

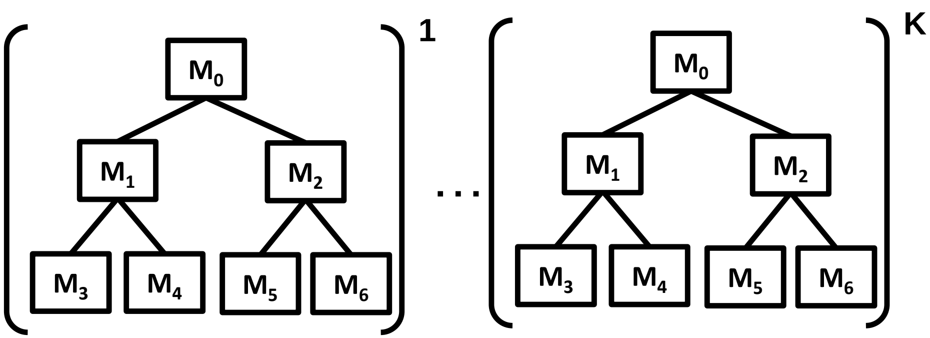
Spatial structure of the anatomy common to all subjects



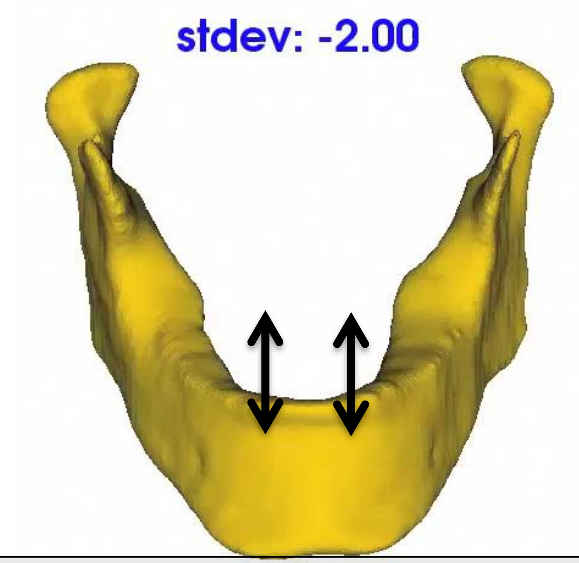


Subject level:

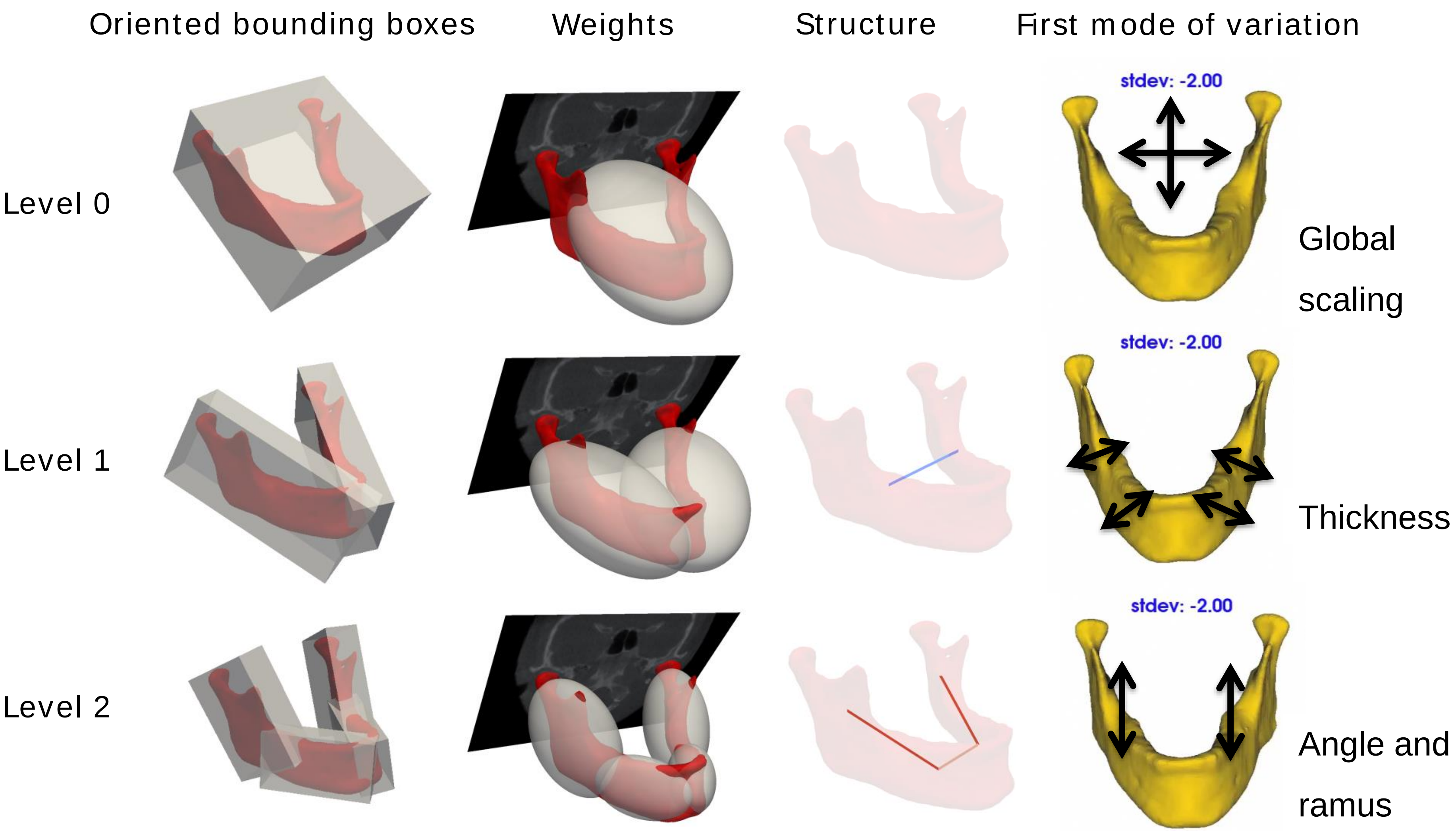
Varying deformation atoms for each subject



Aff(3) valued trees



Hierarchical Estimation of the Variability



47 subjects

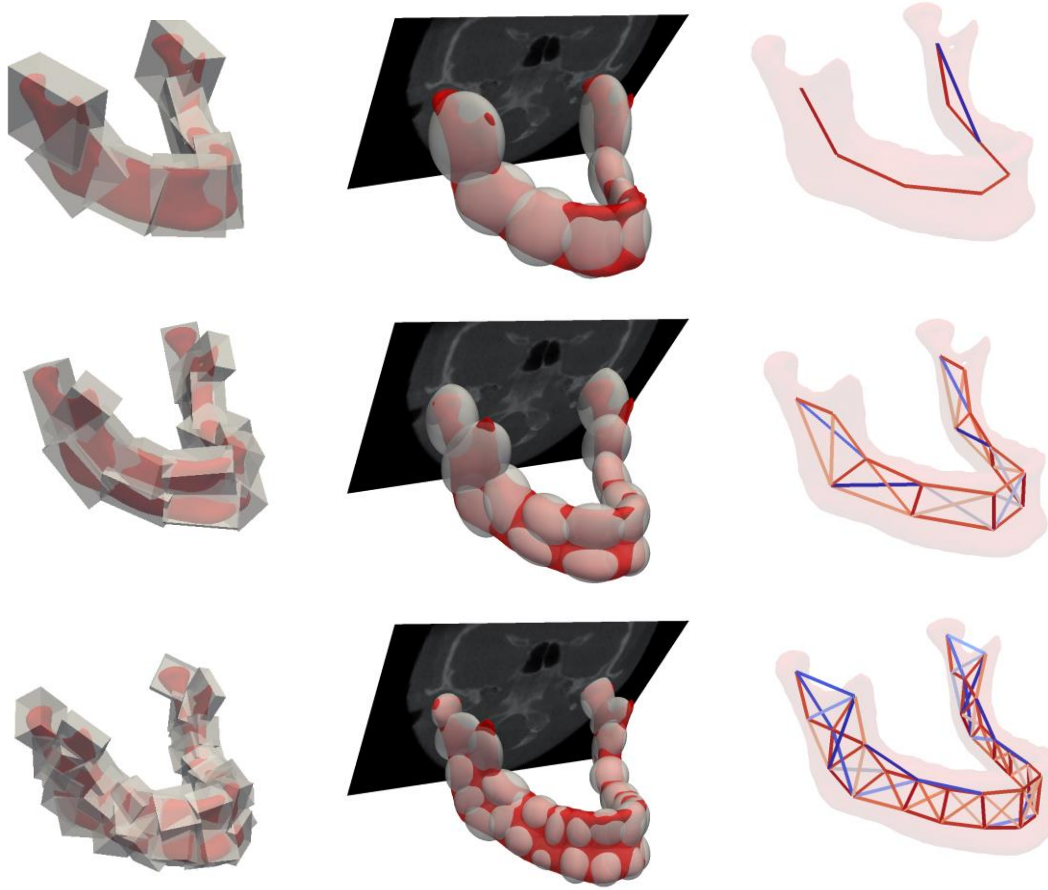
[Seiler, Pennec, Reyes, Medical Image Analysis 16(7):1371-1384, 2012]

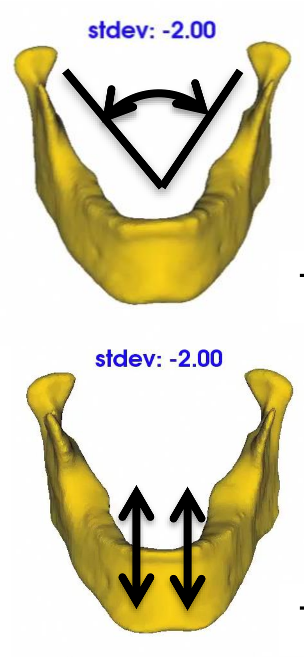
Hierarchical Estimation of the Variability

Level 3

Level 4

Level 5





Two sides

Teeth

47 subjects

[Seiler, Pennec, Reyes, Medical Image Analysis 16(7):1371-1384, 2012]

References for Statistics on Manifolds and Lie Groups

Statistics on Riemannnian manifolds

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