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http://www-sop.inria.fr/asclepios/cours/Peyresq_2019/

Geometric Statistics

Mathematical foundations and applications in computational anatomy

Freely adapted from "Women teaching geometry", in Adelard of Bath translation of Euclid's elements, 1310.

2/ Manifold-valued image processing

Ecole d'été de Peyresq, Jul 1-5 2019

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Manifold-Valued Image Processing

Metric and Affine Geometric Settings for Lie Groups

Parallel Transport to Analyze Longitudinal Deformations

Advances Statistics: CLT & PCA

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- Introduction to diffusion tensor imaging
- Interpolation, filtering, extrapolation of tensors
- Other metrics and applications

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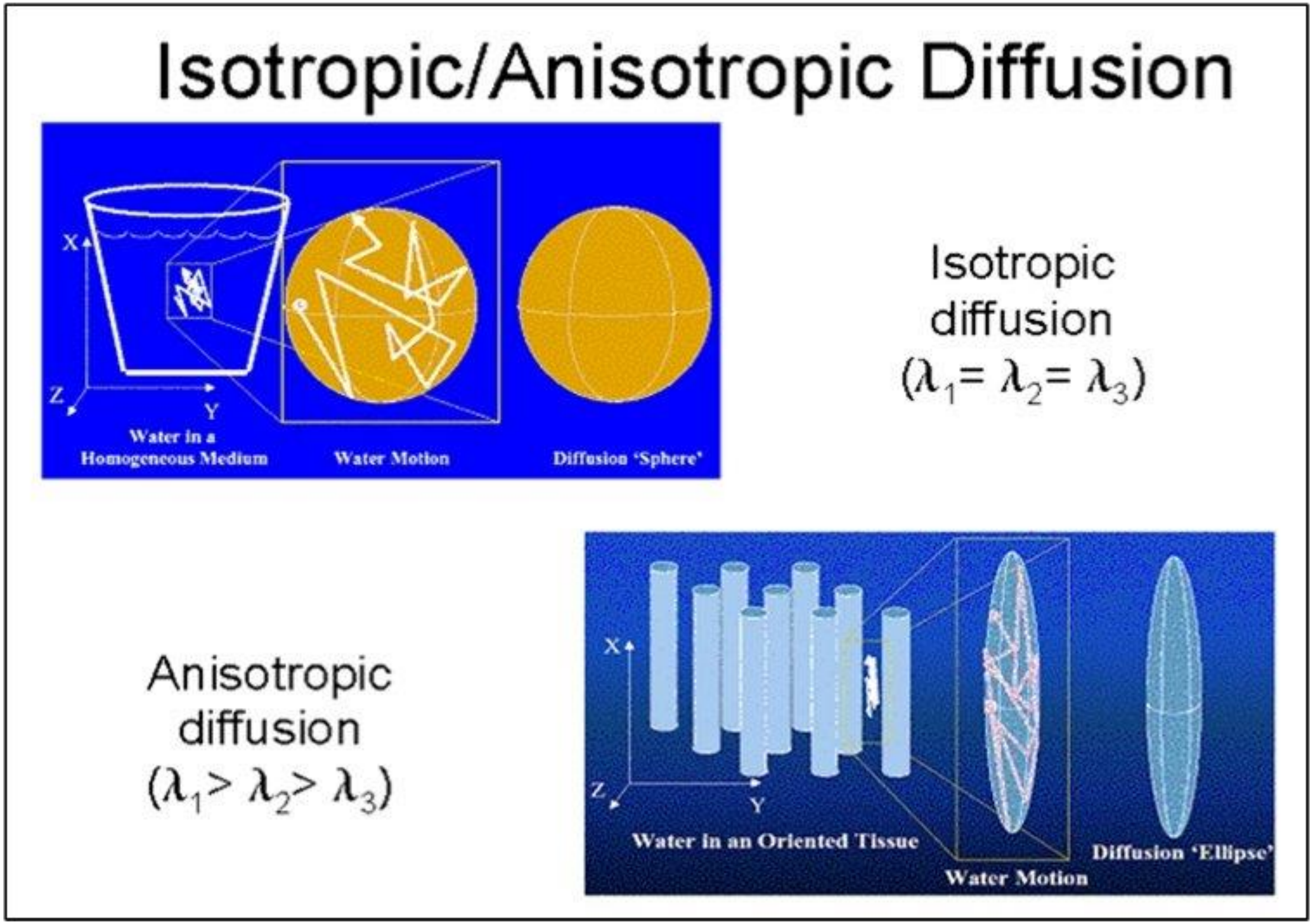
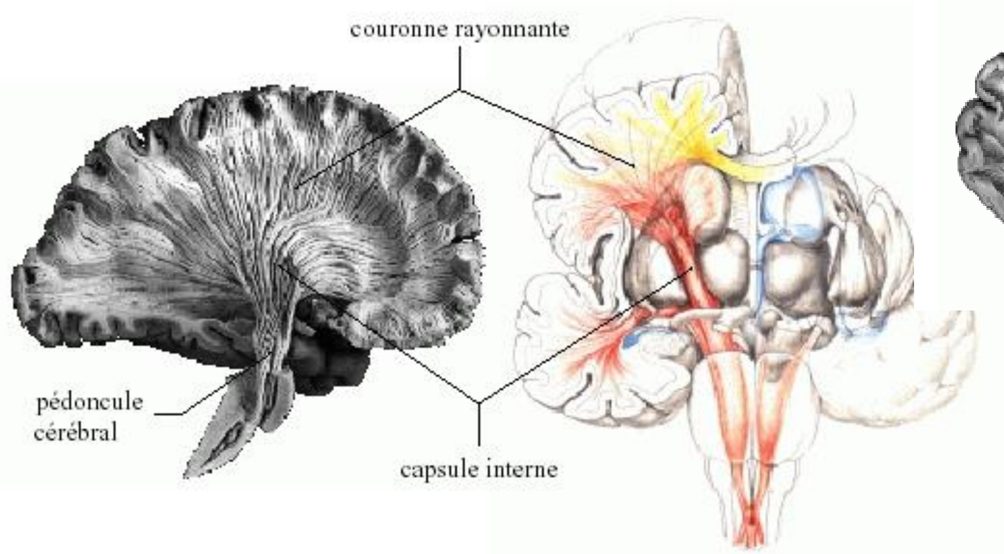
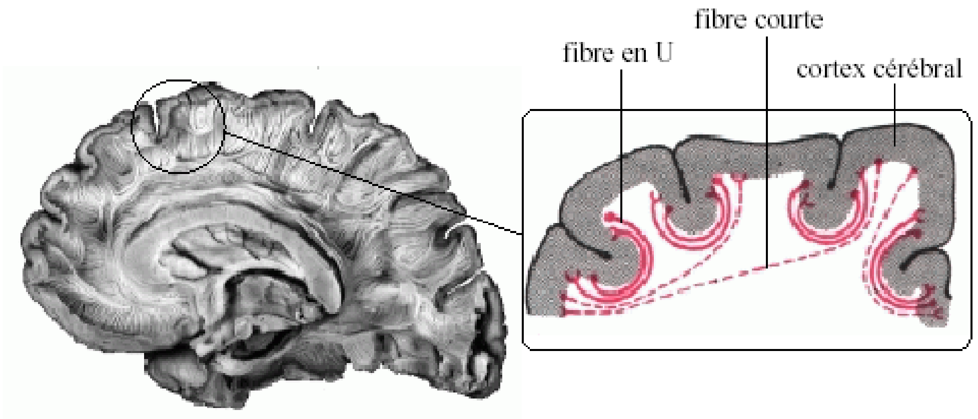
Introduction to diffusion tensor imaging (DTI)



Introduction to diffusion tensor imaging (DTI)

- Classical MRI:
  - white mater, grey matter, CSF
- Diffusion MRI:
  - MR technique born in the mid-80ies (Basser, LeBihan).
  - in vivo imaging of the white mater architecture
  - set of nervous fibers (axons): information highways of the brain!

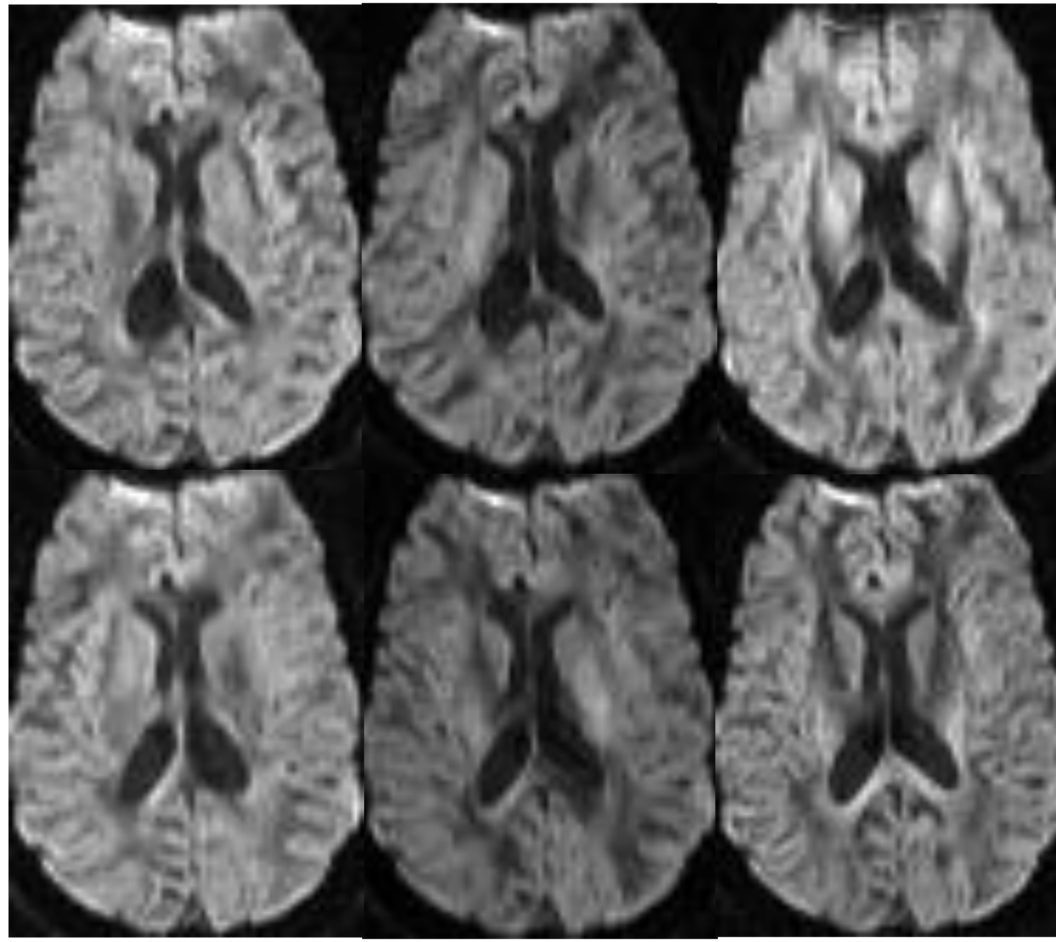




Anatomy of a diffusion MRI

T2 image

\+



6 diffusion weighted images

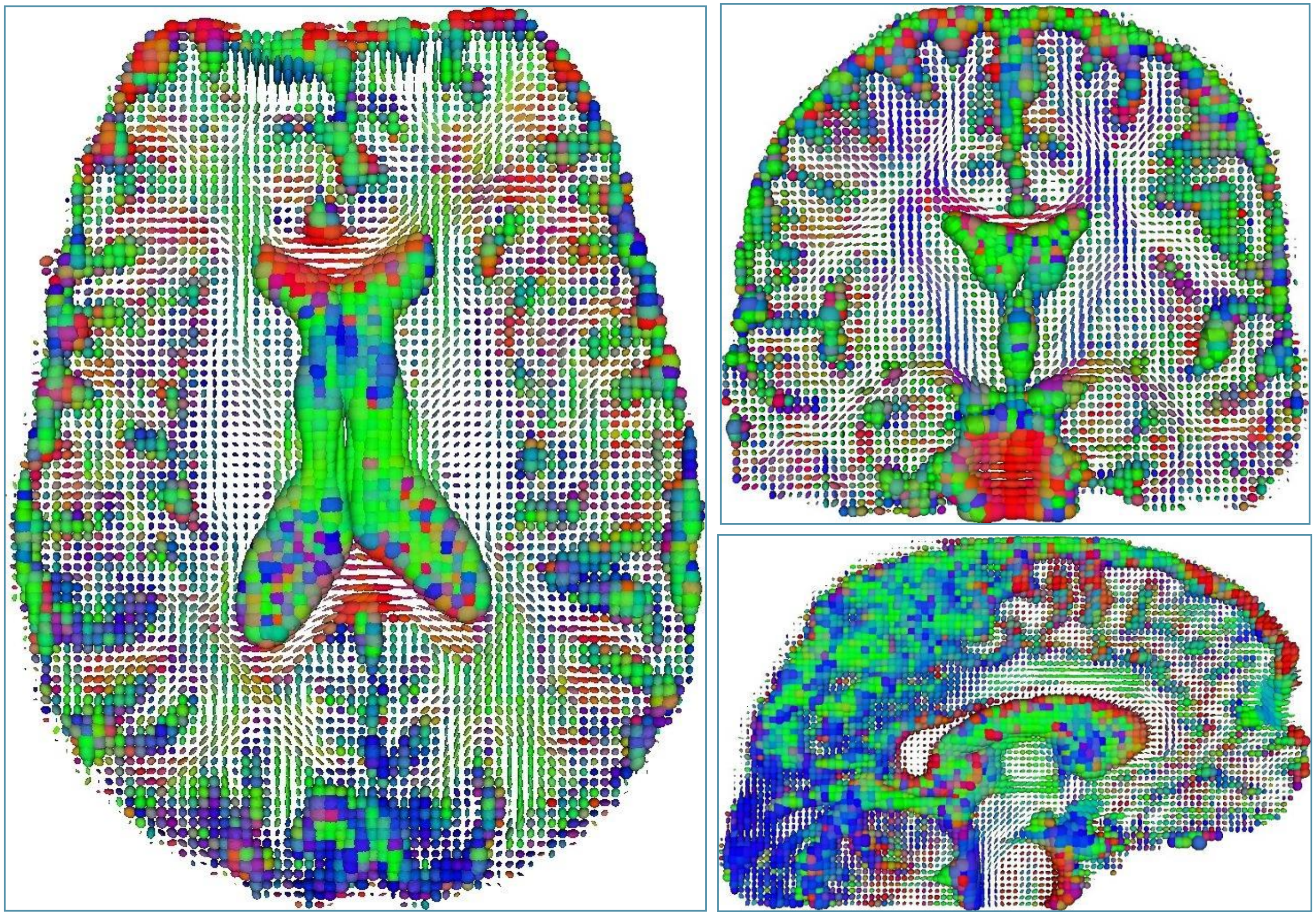
Stejskal & Tanner equation

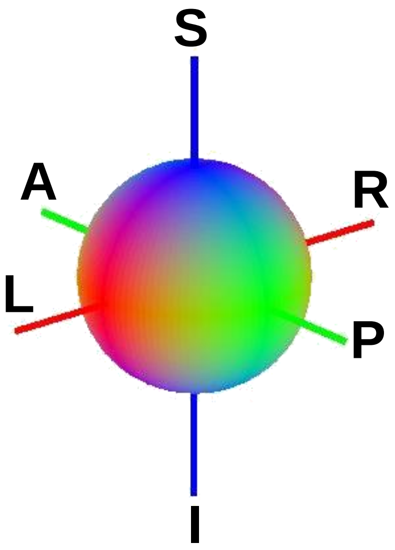
Signal attenuation related to tensor using:

$$S_i = S_0 \, exp(-b.\overrightarrow{g_i}^t.D.\overrightarrow{g_i})$$

- S_i: diffusion weighted images;
- S_0: T2 image;
- g_i: spatial direction of the diffusion gradient;
- b: b-value (related to physical parameters of the acquisition, including field strength and diffusion time);
- D: diffusion tensor;

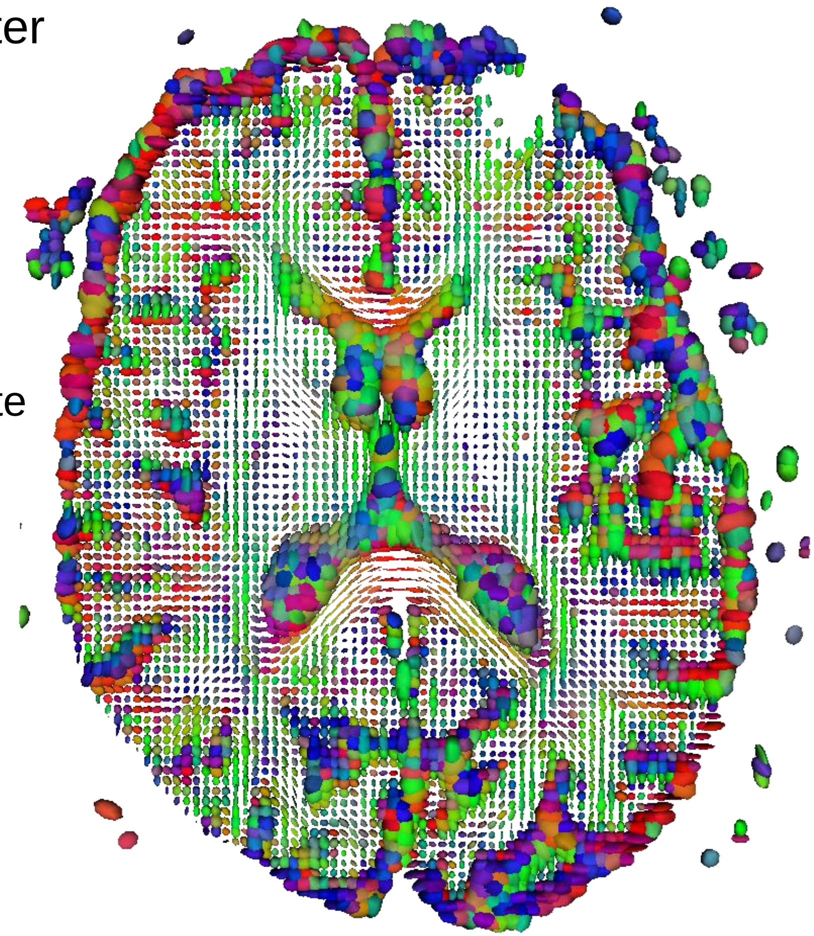
Visualization using ellipsoids





Diffusion Tensor Imaging

Covariance of the Brownian motion of water
-> Architecture of axonal fibers



Diffusion Tensor Field
(slice of a 3D volume)

Symmetric positive definite matrices

- Cone: Convex operations are stable
  - mean, interpolation
  - Null eigenvalues are reachable in a finite time : not physical!
- More complex operations are not
  - PDEs, gradient descent…

SPD-valued image processing

- Clinical images: Very noisy data
- Robust estimation
- Filtering, regularization
- Interpolation / extrapolation

Intrinsic computing on Manifold-valued images?

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GL(n)-Invariant Metrics on SPD matrices

Action of the Linear Group GL$_n$ $$A * \Sigma = A.\Sigma.A^T$$

Invariant metric $$\langle W_1 \mid W_2 \rangle_\Sigma = \langle A W_1 A^t \mid A W_2 A^t \rangle_{A\Sigma A^t} \overset{def}{=} \langle \Sigma^{-1/2} * W_1, \Sigma^{-1/2} * W_2 \rangle_{Id}$$

- Isotropy group at the identity: Rotations
- All rotationally invariant scalar products:

$$\langle W_1 \mid W_2 \rangle_{Id} \overset{def}{=} \mathrm{Tr}\left(W_1^T W_2\right) + \beta \, \mathrm{Tr}(W_1).\mathrm{Tr}(W_2) \quad (\beta > \text{-}1/\mathrm{n})$$

- Geodesics at Id $\quad \Gamma_{Id,W}(t) = \exp(tW)$
- Exponential map $\quad Exp_\Sigma(\overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(\Sigma^{-1/2}.\overrightarrow{\Sigma\Psi}.\Sigma^{-1/2})\Sigma^{1/2}$
- Log map $\quad \overrightarrow{\Sigma\Psi} = Log_\Sigma(\Psi) = \Sigma^{1/2} \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2})\Sigma^{1/2}$
- Distance $\quad \boxed{dist(\Sigma, \Psi)^2 = \langle \overrightarrow{\Sigma\Psi} \mid \overrightarrow{\Sigma\Psi} \rangle_\Sigma = \left\| \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2}) \right\|_{Id}^2}$

GL(n)-Invariant Metrics on Tensors

$$\|W\|_{\Sigma}^{2} = \mathrm{Tr}\left(W.\Sigma^{-1}W\Sigma^{-1}\right) + \beta\,\mathrm{Tr}\left(W\Sigma^{-1}\right)^{2} \quad (\beta > \text{-}1/\mathrm{n})$$

Space of Gaussian distributions (Information geometry) (β=0)

- Fisher information metric [Burbea & Rao J. Multivar Anal 12 1982, Skovgaard Scand J. Stat 11 1984, Calvo & Oller Stat & Dec. 9 1991]
- Tensor segmentation [Lenglet RR04 & JMIV 25(3) 2006]

Affine-invariant metrics for DTI processing (β=0)

- [Pennec, Fillard, Ayache, IJCV 66(1), Jan 2006 / INRIA RR-5255, 2004]
- PGA on tensors [Fletcher & Joshi CVMIA04, SigPro 87(2) 2007]

Geometric means (β=0)

- Covariance matrices in computer vision [Forstner TechReport 1999]
- Math. properties [Moakher SIAM J. Matrix Anal App 26(3) 2004]
- Geodesic Anisotropy [Batchelor MRM 53 2005]

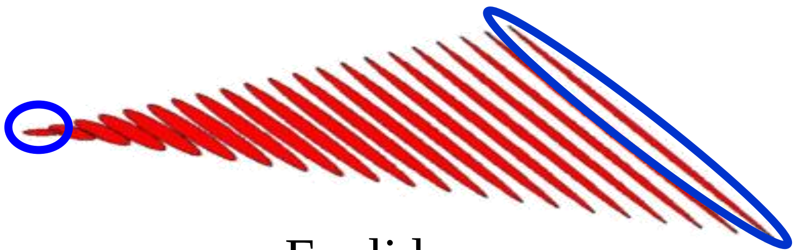
Homogeneous Embedding (β=-1/(n+1))

- [Lovric & Min-Oo, J. Multivar Anal 74(1), 2000]

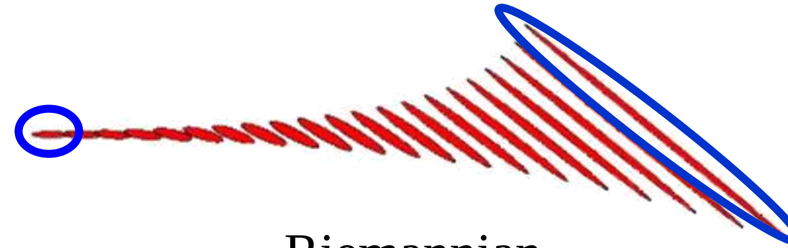
Tensor interpolation

Geodesic walking in 1D

$$\Sigma(t) = \exp_{\Sigma_1}(t\overrightarrow{\Sigma_1\Sigma_2})$$



Euclidean

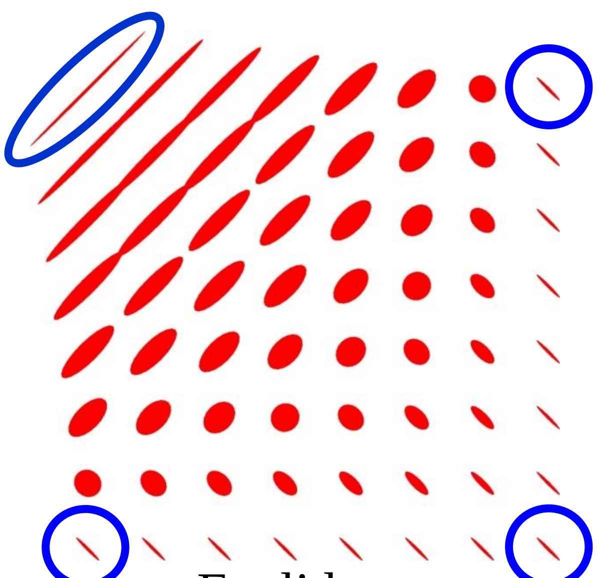


Riemannian

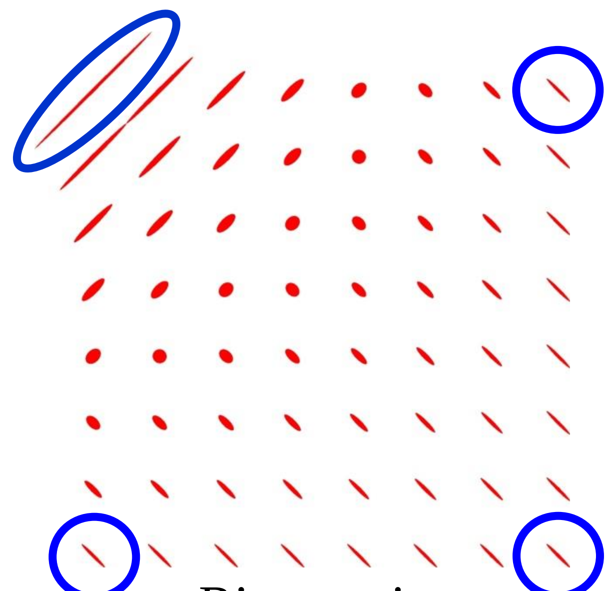
Bi-linear, tri-linear interpolation?

Rephrase as weighted mean

$$\Sigma(x) = \min_{\Sigma} \sum w_i(x)\ dist(\Sigma, \Sigma_i)^2$$



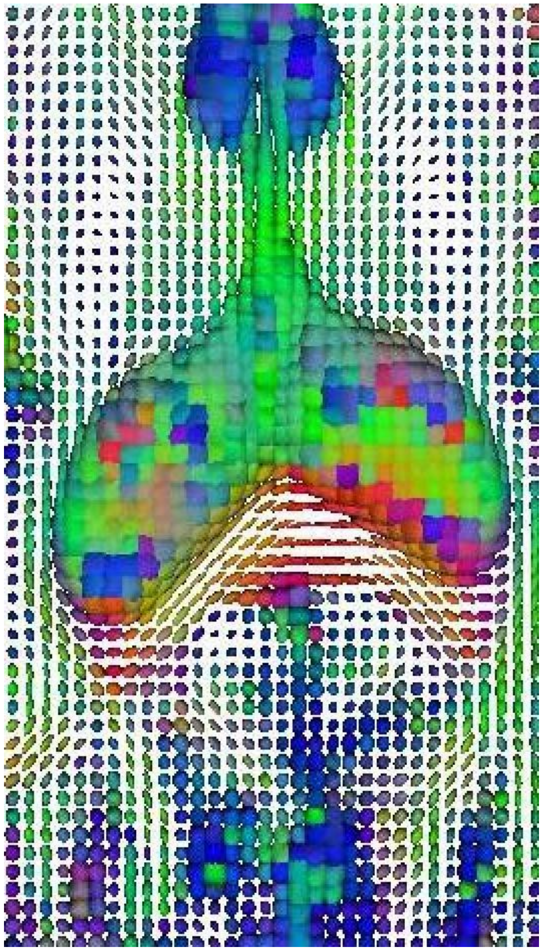
Euclidean



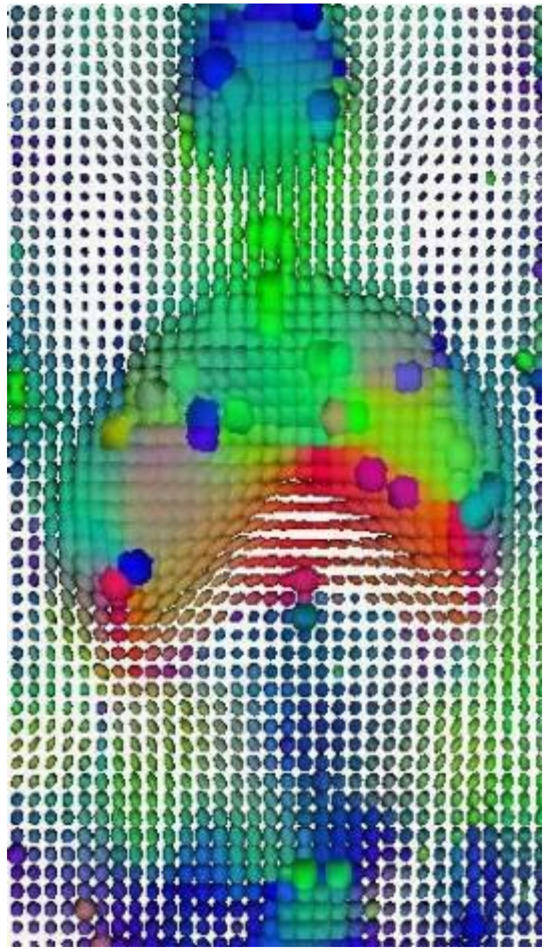
Riemannian

Gaussian filtering: Gaussian weighted mean

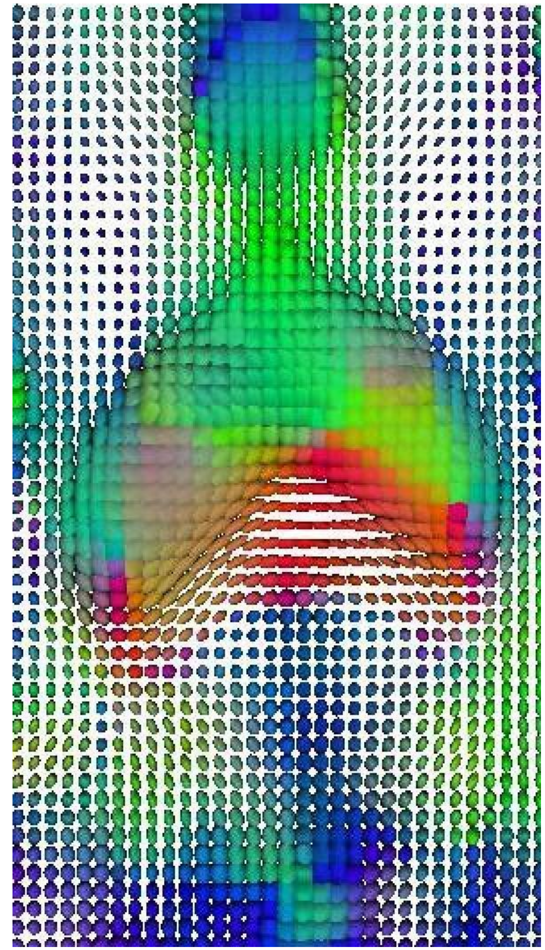
$$\Sigma(x) = \min \sum_{i=1}^{n} G_{\sigma}(x - x_i)\ dist(\Sigma, \Sigma_i)^2$$



Raw



Coefficients σ=2



Riemann σ=2

PDE for filtering and diffusion

Harmonic regularization

$$C(\Sigma) = \int_{\Omega} \left\| \nabla \Sigma(x) \right\|^2_{\Sigma(x)} dx$$

- Gradient = manifold Laplacian

$$\Delta\Sigma(x) = \sum_{i=x,y,z} \partial_i^2 \Sigma - \sum_{i=x,y,z} \left(\partial_i \Sigma\right) \Sigma^{(-1)} \left(\partial_i \Sigma\right) = \sum_u \frac{\overrightarrow{\Sigma(x)\Sigma(x+u)}}{\|u\|^2} + O\left(\|u\|^2\right)$$

- Trivial intrinsic numerical schemes with exponential maps!

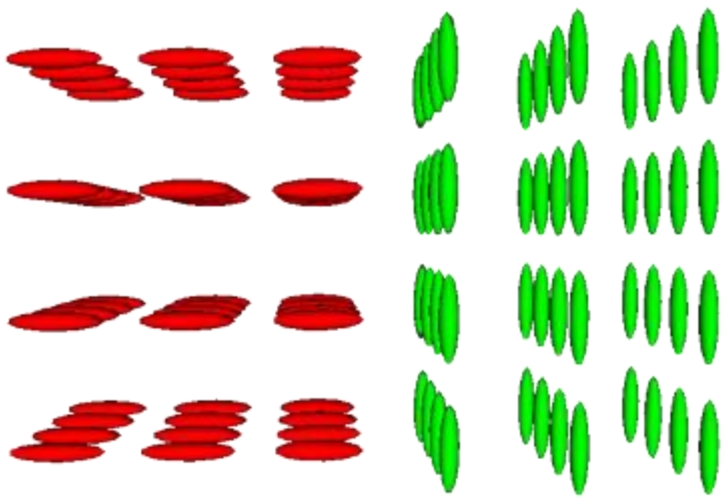
- Geodesic marching $\Sigma_{t+1}(x) = \exp_{\Sigma_t(x)}\left(-\varepsilon \nabla C(\Sigma)(x)\right)$

Anisotropic diffusion

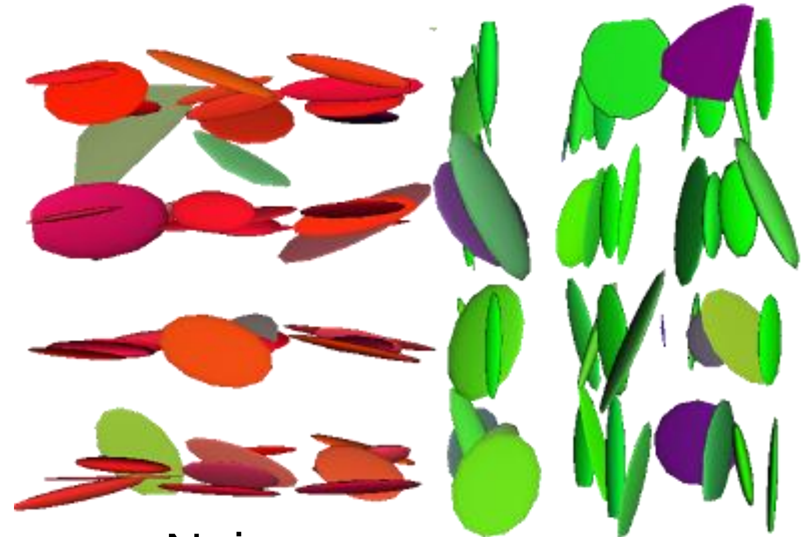
- Perona-Malik 90 / Gerig 92 $\Delta_w \Sigma(x) = \sum_u w\left(\left\|\partial_u \Sigma(x)\right\|_{\Sigma(x)}\right) \Delta_u \Sigma(x)$
- Robust functions $\mathrm{Reg}(\Sigma) = \int \Phi\left(\left\|\nabla\Sigma(x)\right\|^2_{\Sigma(x)}\right) dx$

[Pennec, Fillard, Arsigny, IJCV 66(1), 2005, ISBI 2006]

Anisotropic filtering

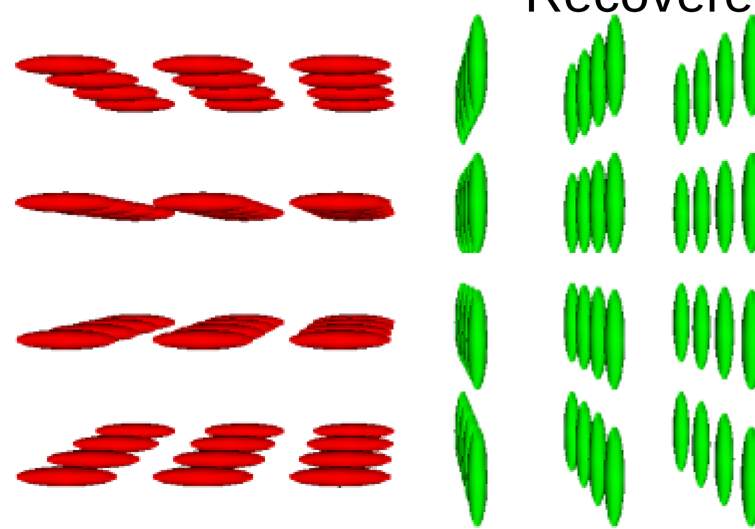


Initial



Noisy

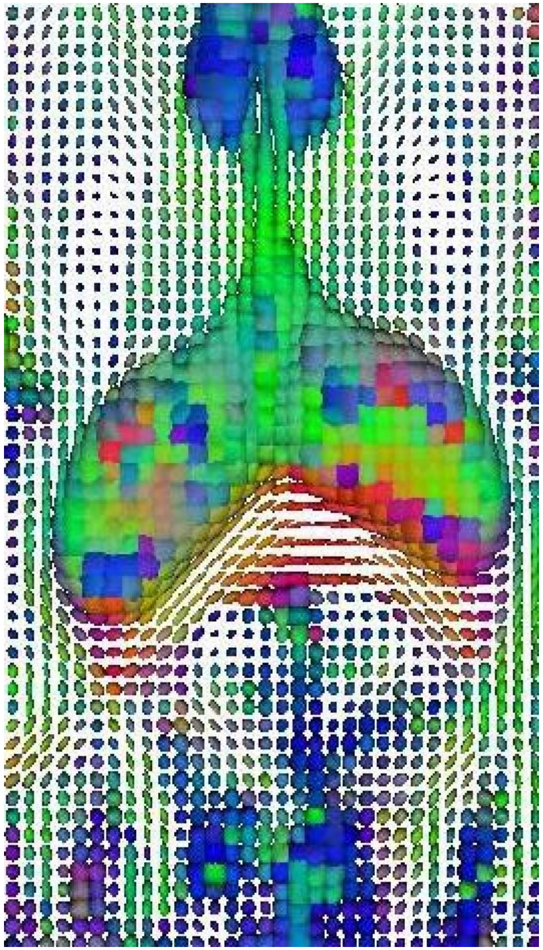
Recovered



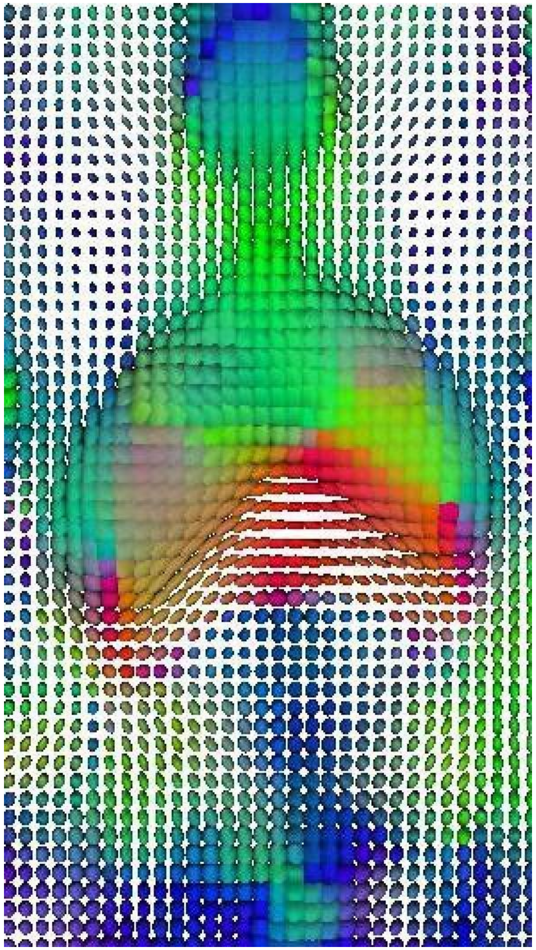
Anisotropic filtering

$$\hat{\Delta}\Sigma(x) = \sum_u w\left(\|\partial_u \Sigma(x)\|_{\Sigma(x)}\right) \Delta_u \Sigma(x) \quad \text{with} \quad w(t) = \exp(-t^2/\kappa^2)$$

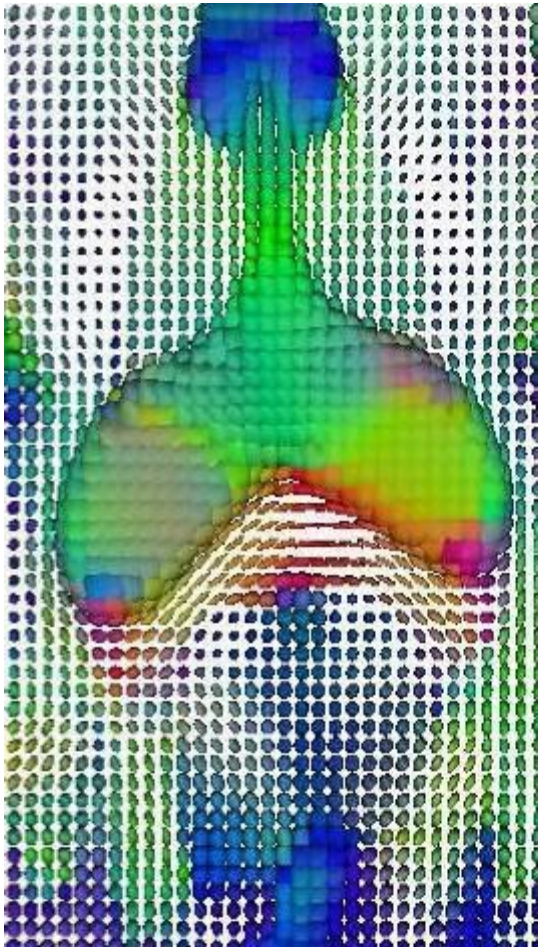
$$\Delta_u \Sigma(x) = \partial_u^2 \Sigma - \left(\partial_u \Sigma\right) \Sigma^{(-1)} \left(\partial_u \Sigma\right) \approx \overrightarrow{\Sigma(x)\Sigma(x+u)} / \|u\|^2$$



Raw



Riemann Gaussian

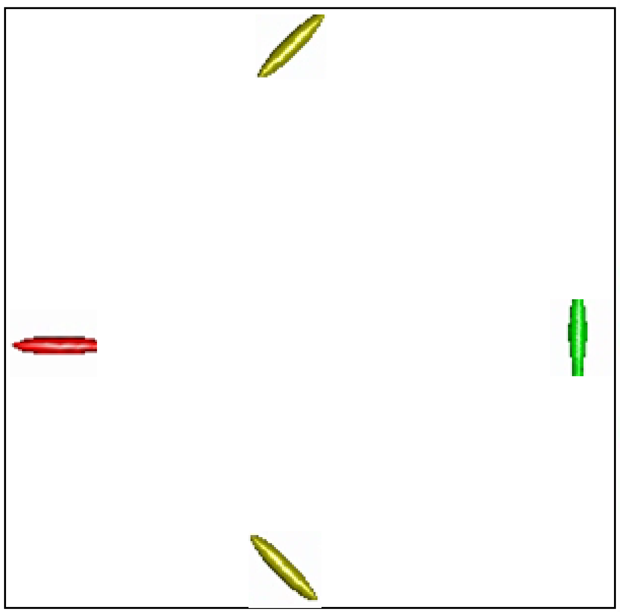


Riemann anisotropic

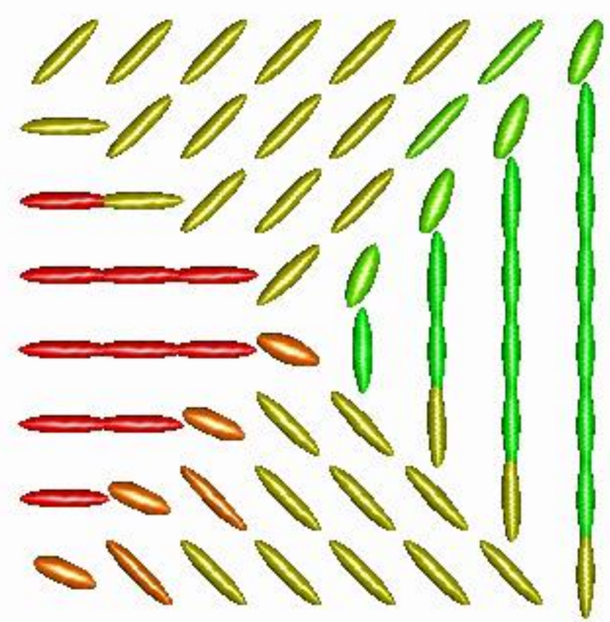
Extrapolation by Diffusion

$$C(\Sigma) = \frac{1}{2}\int_{\Omega} \sum_{i=1}^{n} G_{\sigma}(x - x_i)\, dist(\Sigma(x), \Sigma_i)^2\, dx + \frac{\lambda}{2}\int_{\Omega} \left\| \nabla \Sigma(x) \right\|^2_{\Sigma(x)}$$

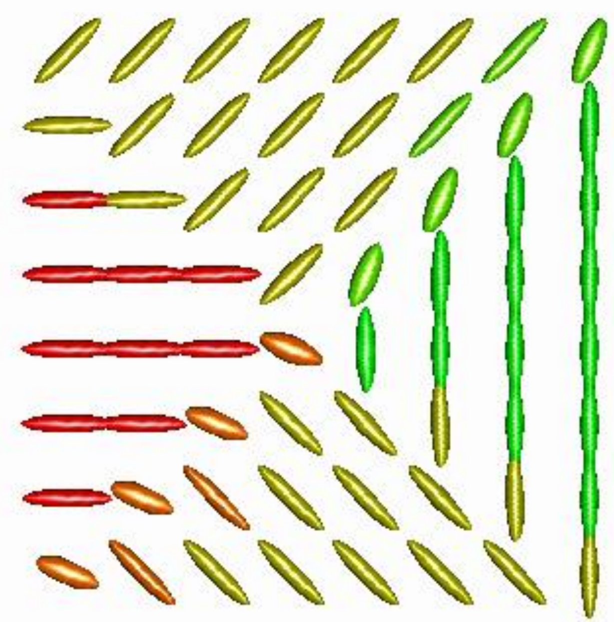
$$\nabla C(\Sigma)(x) = -\sum_{i=1}^{n} G_{\sigma}(x - x_i)\, \overrightarrow{\Sigma(x)\Sigma_i} - \lambda (\Delta \Sigma)(x)$$



Original tensors



Diffusion λ=0.01



Diffusion λ=∞

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Log Euclidean Metric on Tensors

Exp/Log: global diffeomorphism Tensors/sym. matrices

- Vector space structure carried from the tangent space to the manifold
  - Log. product $\Sigma_1 \otimes \Sigma_2 \equiv \exp(\log(\Sigma_1) + \log(\Sigma_2))$
  - Log scalar product $\alpha \bullet \Sigma \equiv \exp(\alpha \log(\Sigma)) = \Sigma^\alpha$
  - Bi-invariant metric $\boxed{dist(\Sigma_1, \Sigma_2)^2 \equiv \|\log(\Sigma_1) - \log(\Sigma_2)\|^2}$

Properties

- Invariance by the action of similarity transformations only
- Very simple algorithmic framework

[Arsigny, MICCAI 2005 & MRM 56(2), 2006]

Riemannian Frameworks on tensors

Affine-invariant Metric (Curved space – Hadamard)

- Dot product $\langle V \mid W \rangle_\Sigma = \langle AVA^T \mid AWA^T \rangle_{A\Sigma A^T} = \langle \Sigma^{-1/2} V \Sigma^{-1/2} \mid \Sigma^{-1/2} W \Sigma^{-1/2} \rangle_{Id}$
- Geodesics $Exp_\Sigma(t.\overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(t.\Sigma^{-1/2}.\overrightarrow{\Sigma\Psi}.\Sigma^{-1/2})\Sigma^{1/2}$
- Distance $\boxed{\mathrm{dist}(\Sigma,\Psi)^2 = \langle \overrightarrow{\Sigma\Psi} \mid \overrightarrow{\Sigma\Psi} \rangle_\Sigma = \left\| \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2}) \right\|_{L_2}^2}$

[Pennec, Fillard, Ayache, IJCV 66(1), 2006, Lenglet JMIV'06, etc]

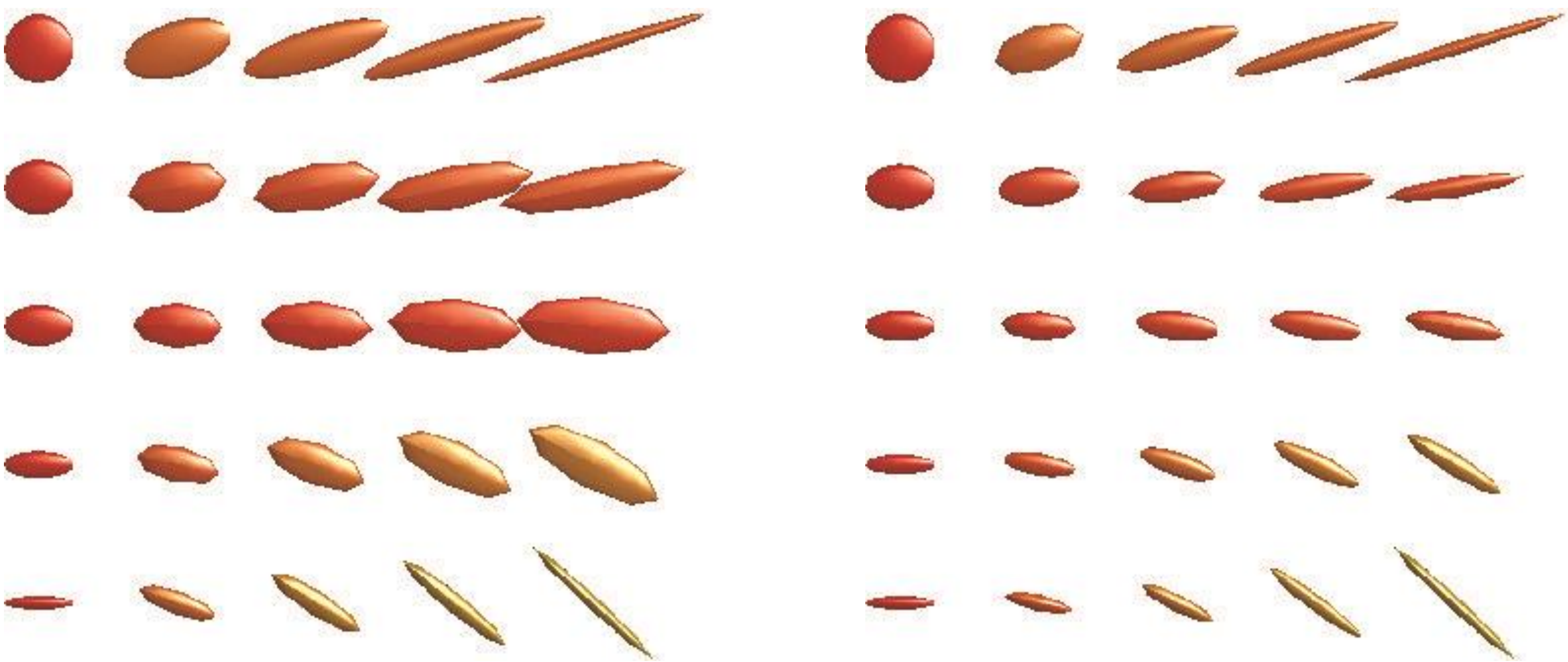
Log-Euclidean similarity invariant metric (vector space)

- Transport Euclidean structure through matrix exponential
- Dot product $\langle V \mid W \rangle_\Sigma = \langle \partial_V \log(\Sigma) \mid \partial_W \log(\Sigma) \rangle_{Id}$
- Geodesics $Exp_\Sigma(t.\overrightarrow{\Sigma\Psi}) = \exp(\log(\Sigma) + t.\partial_{\overrightarrow{\Sigma\Psi}} \log(\Sigma))$
- Distance $\boxed{\mathrm{dist}(\Sigma_1,\Sigma_2)^2 \equiv \left\| \log(\Sigma_1) - \log(\Sigma_2) \right\|^2}$

[Arsigny, Pennec, Fillard, Ayache, SIAM'06, MRM'06]

Log Euclidean vs Affine invariant

- Means are geometric (vs arithmetic for Euclidean)
- Log Euclidean slightly more anisotropic
- Speedup ratio: 7 (aniso. filtering) to >50 (interp.)



Euclidean

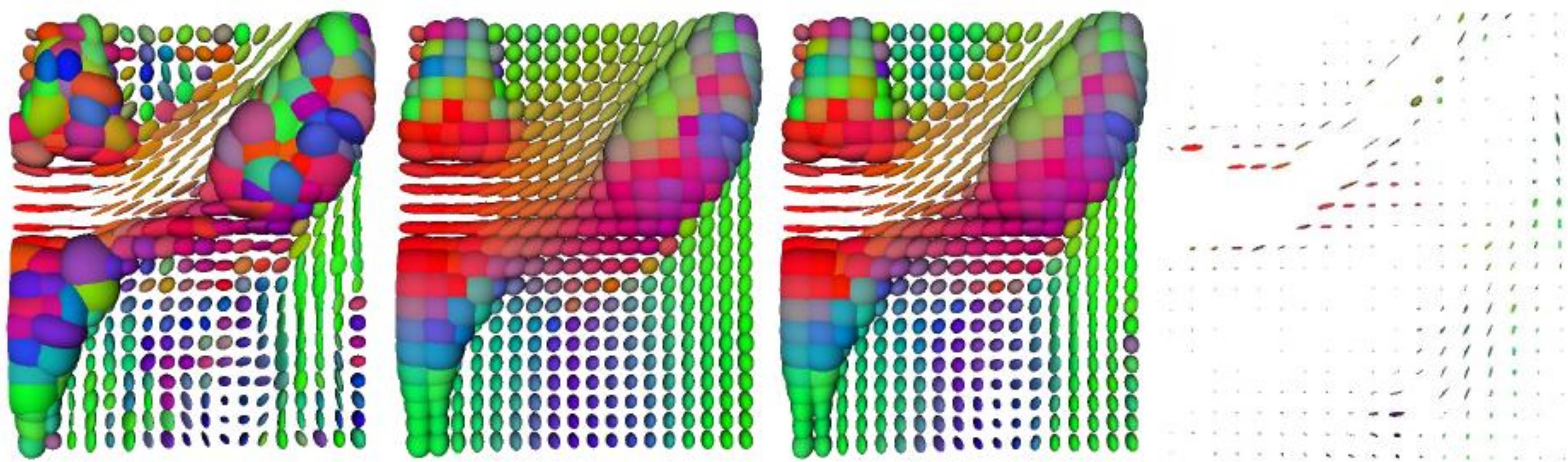
Affine invariant

Log-Euclidean

Log Euclidean vs Affine invariant

Real DTI images: anisotropic filtering

- Difference is not significant
- Speedup of a factor 7 for Log-Euclidean



Original   Euclidean   Log-Euclidean   Diff. LE/affine (x100)

Some other metrics on tensors

Square root metrics

- Cholesky [Wang Vemuri et al, IPMI'03, TMI 23(8) 2004.]
- Size and shape space [Dryden, Koloydenko & Zhou, 2008]
- Power Euclidean [Dryden & Pennec, arXiv 2011]

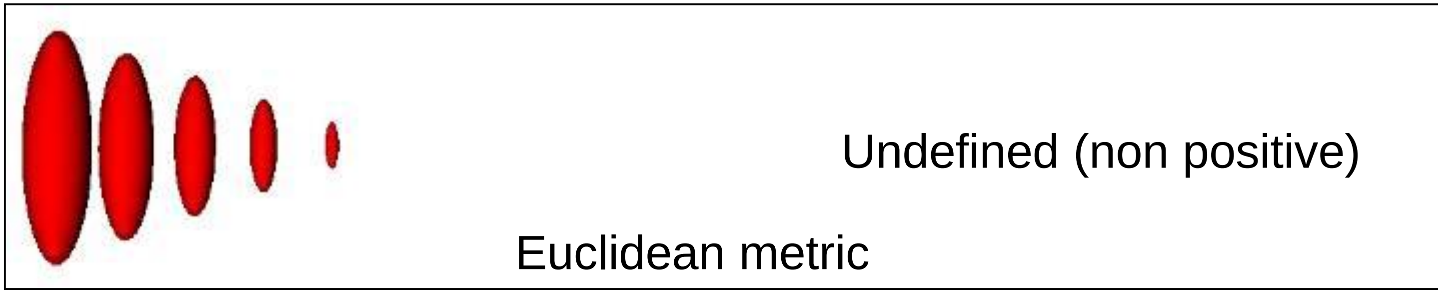
Non Riemannian distances

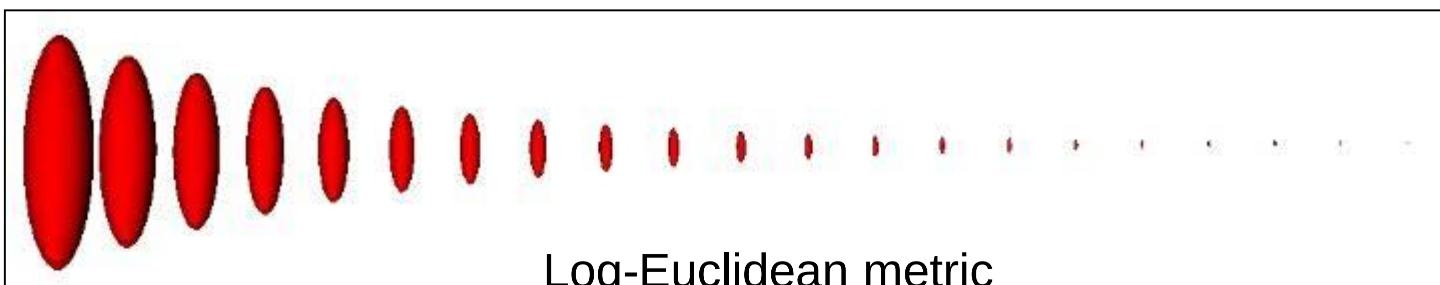
- J-Divergence [Wang & Vemuri, TMI 24(10), 2005]
- Geodesic Loxodromes [Kindlmann et al. MICCAI 2007]

4th order tensors

- [Gosh, Descoteau & Deriche MICCAI'08]

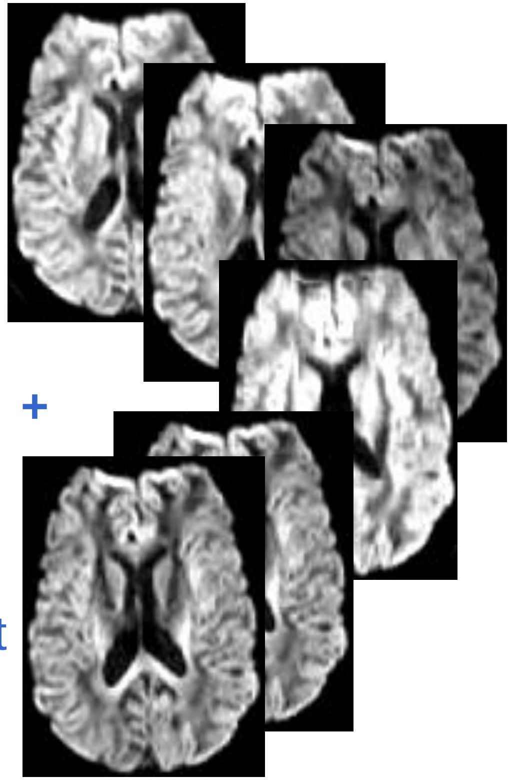
Geodesic shooting in tensors spaces





DTI Estimation from DWI

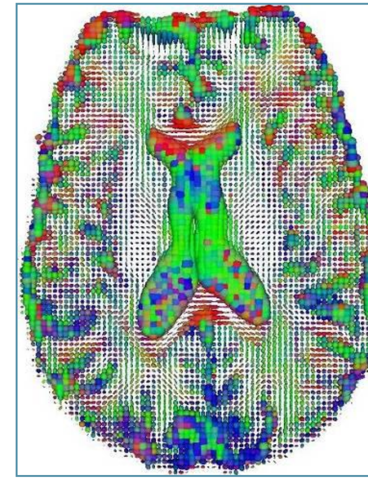
+



Baseline + at least 6 DWI

Stejskal & Tanner diffusion equation

$$\boxed{S_i = S_0 \exp\left(-b g_i^T D g_i\right)}$$



Diffusion Tensor Field

Noise is Gaussian in complex DW signal

- Rician = amplitude of complex Gaussian
- LSQ on Rician noise = bias for low SNRs [Sijbers, TMI 1998]

Estimation / Regularization on complex DWI:

- Anisotropic diffusion on Choleski factors [Wang & Vemuri, TMI'04]

Estimation with a Rician noise

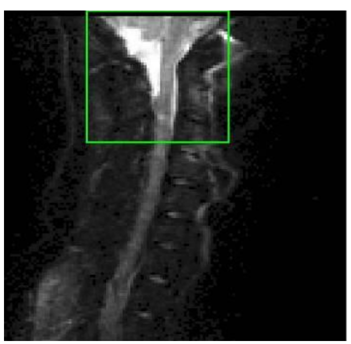
- Smoothing DWI before estimation [Basu & Fletcher, MICCAI 2006]
- ML (MMSE) [Aja-Fernández et al, TMI 2008]
- MAP with log-Euclidean prior [Fillard et al., ISBI 2006, TMI 2007]

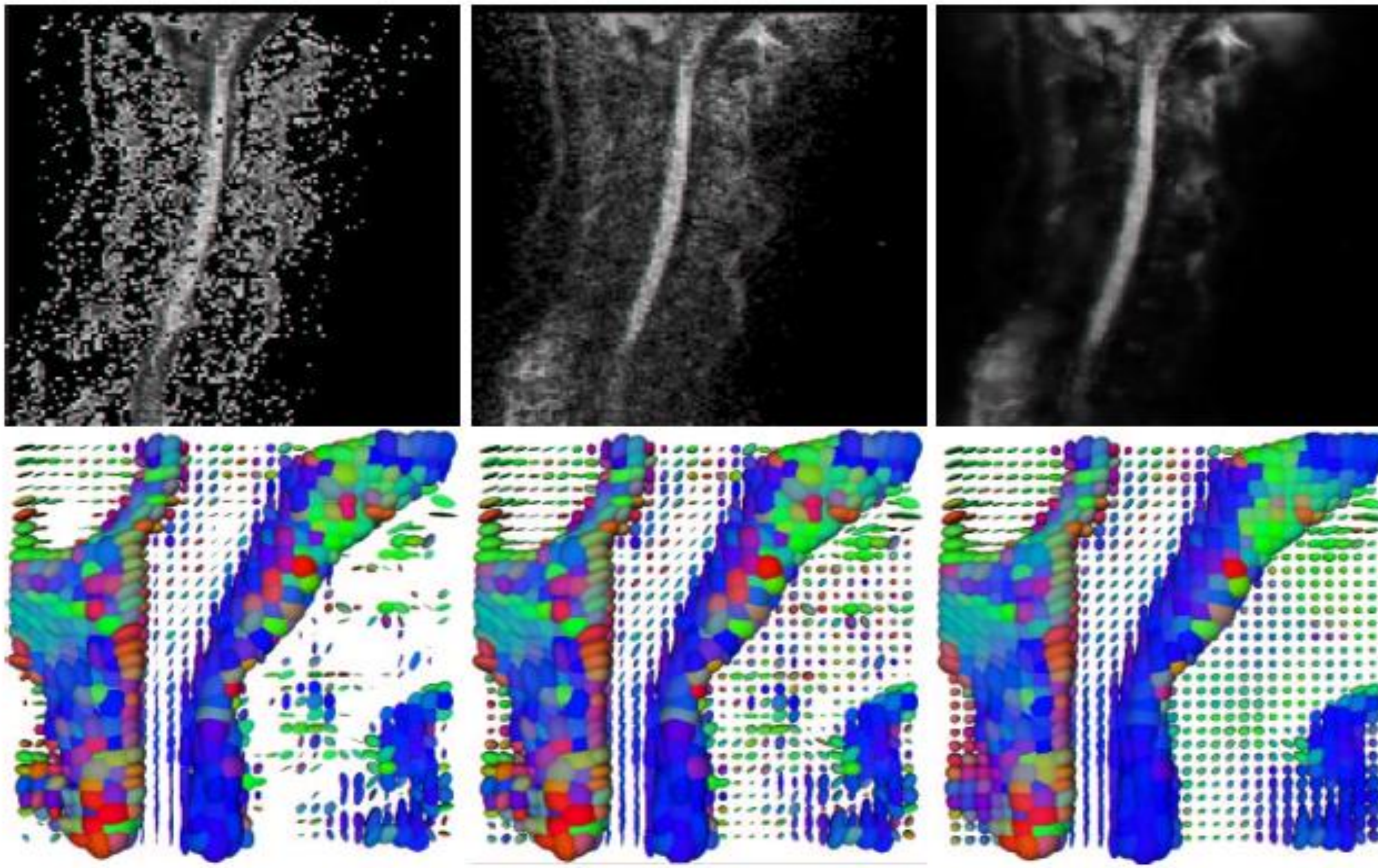
Rician MAP estimation with Riemannian spatial prior

$$MAP(\Sigma) = -\sum_{i=1}^{N} \int \log\left(\frac{\hat{S}_i}{\sigma^2} \exp\left(-\frac{\hat{S}_i^{\,2} + S_i(\Sigma)^2}{2\sigma^2} \right) I_0\left(\frac{S_i(\Sigma)\hat{S}_i}{\sigma^2} \right) \right) .dx + \int \Phi\left(\|\nabla\Sigma(x)\|^2_{\Sigma(x)} \right) dx$$

FA

Estimated tensors

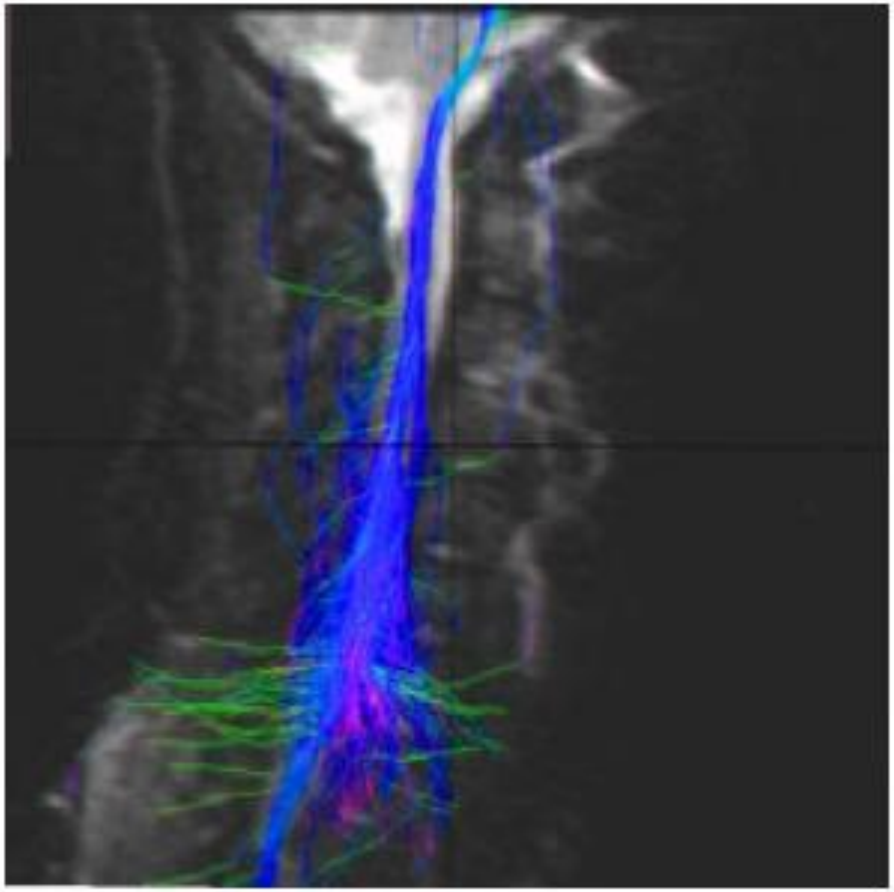




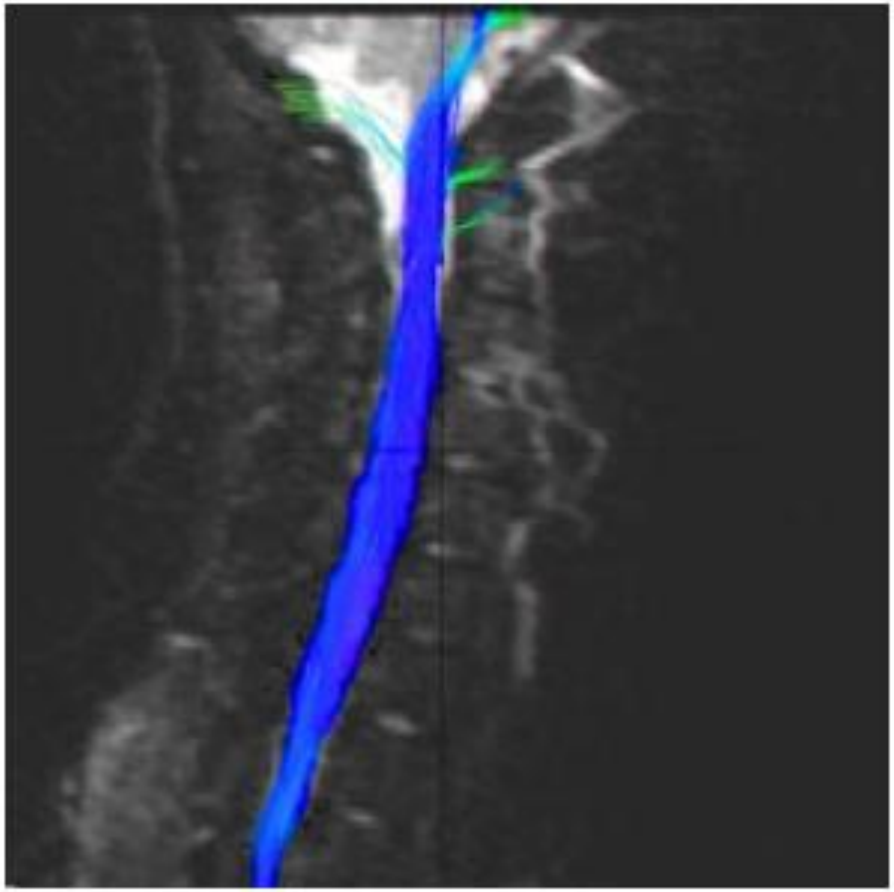
Standard ML Rician MAP Rician

[Fillard, Arsigny, Pennec, Ayache ISBI'06, TMI 26(11) 2007]

Clinical DTI of the spinal cord: fiber tracking

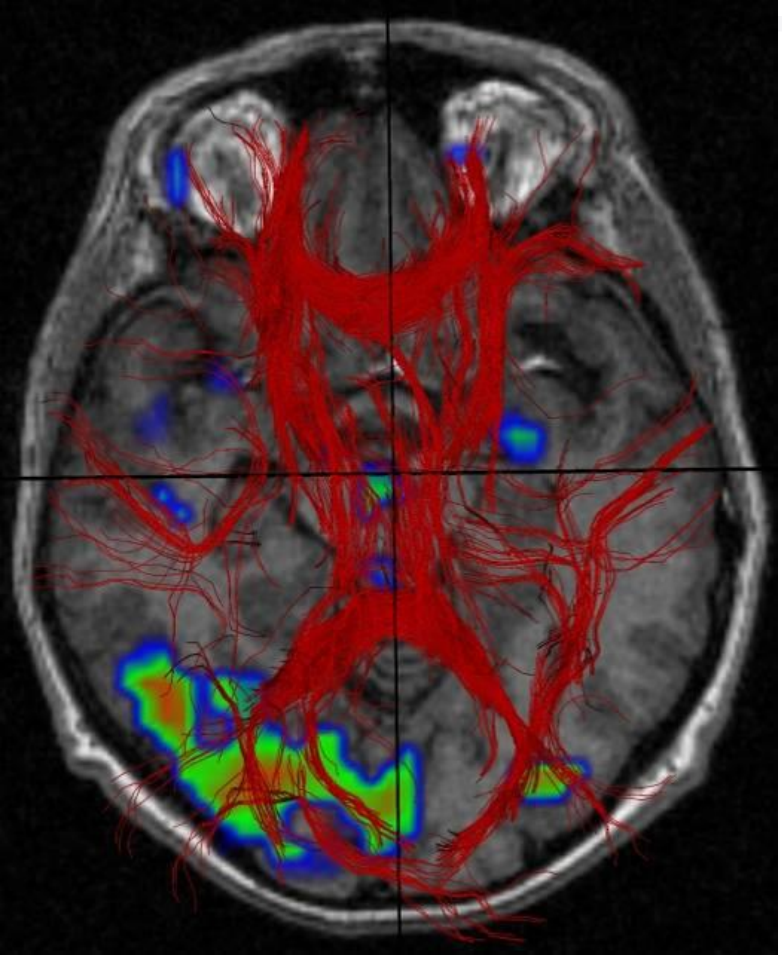


Standard

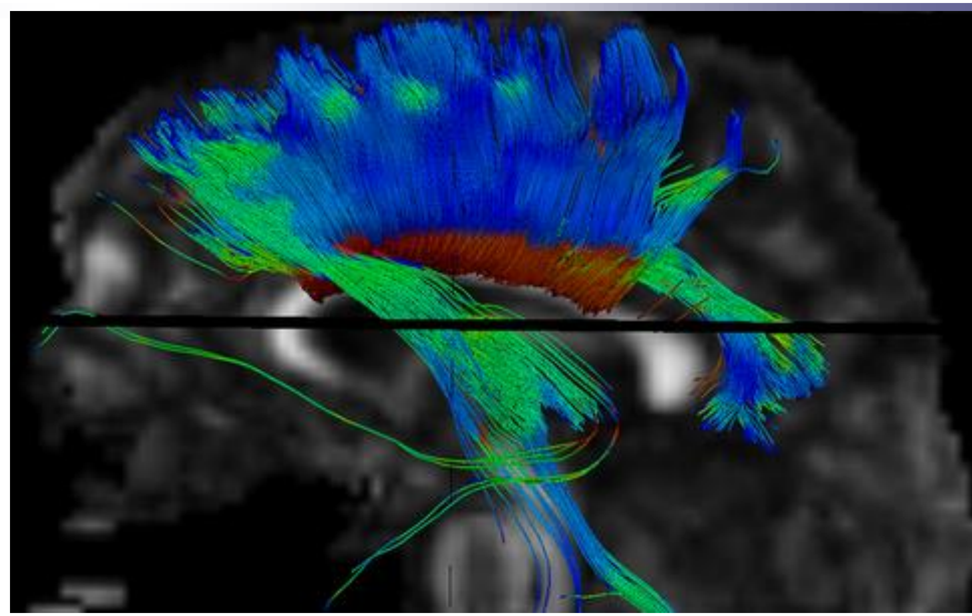


MAP Rician

[Fillard, Arsigny, Pennec, Ayache ISBI'06, TMI 26(11) 2007]



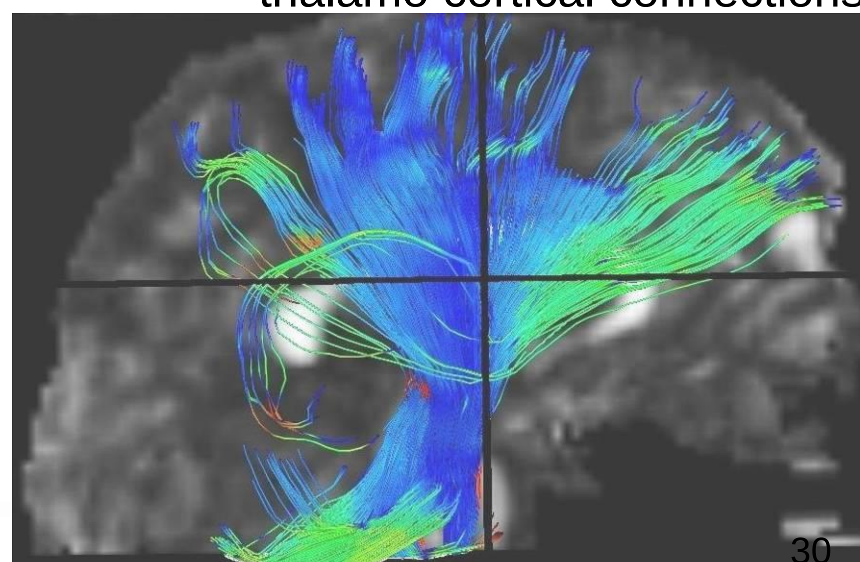
T1 + Activation map + fibers

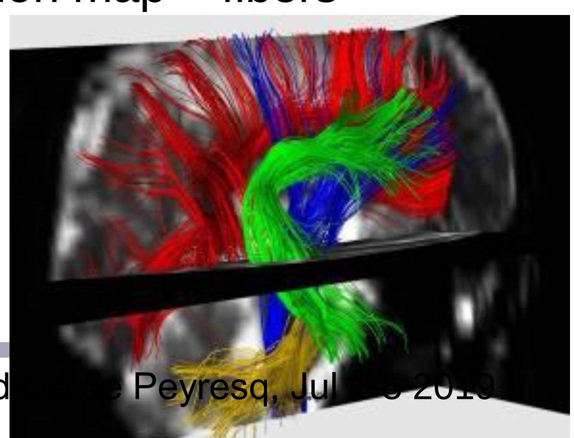


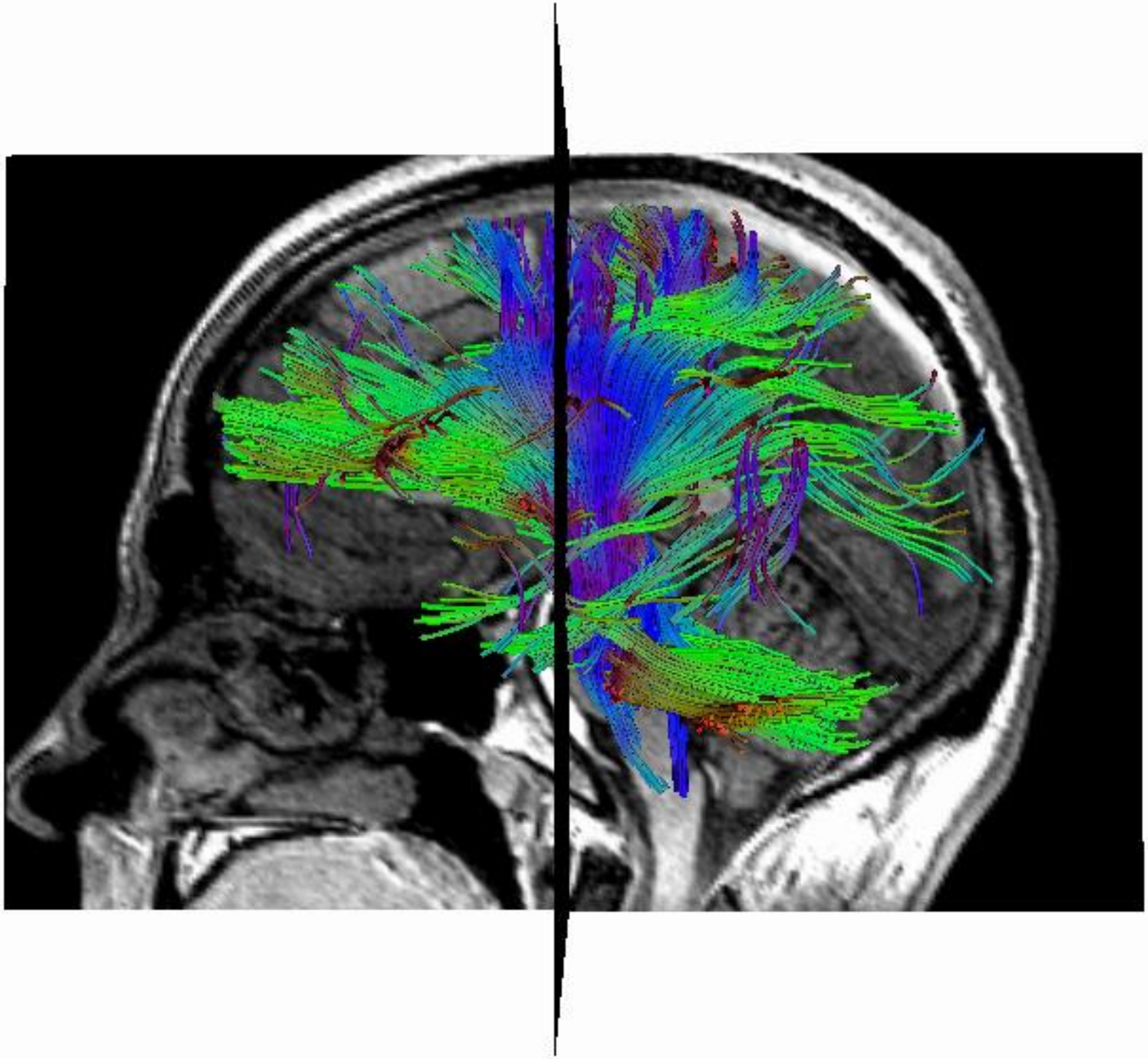
Corpus callosum + cingulum

Courtesy of P. Fillard using MedINRIA

Corticospinal tract and thalamo cortical connections







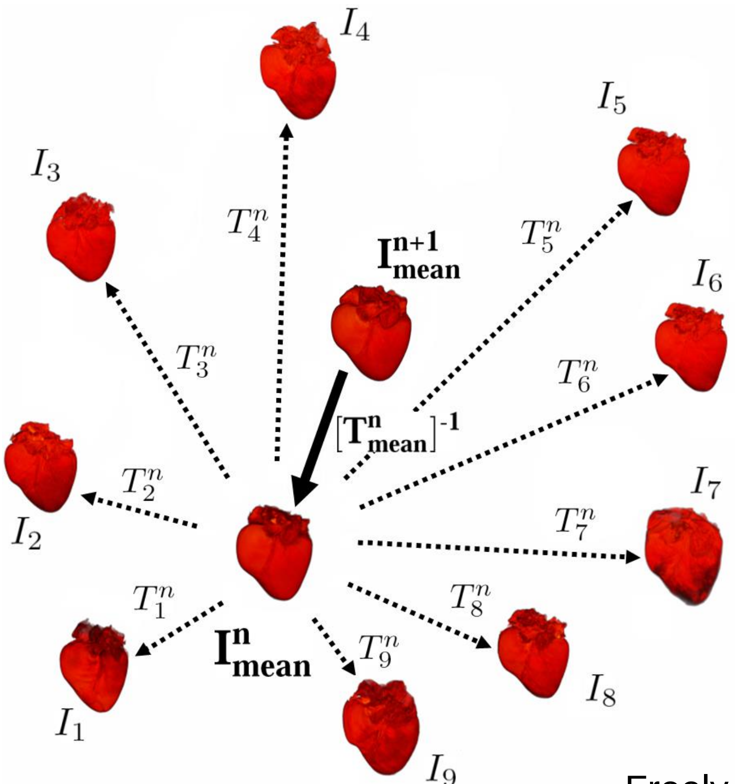
A Statistical Atlas of the Cardiac Fiber Structure

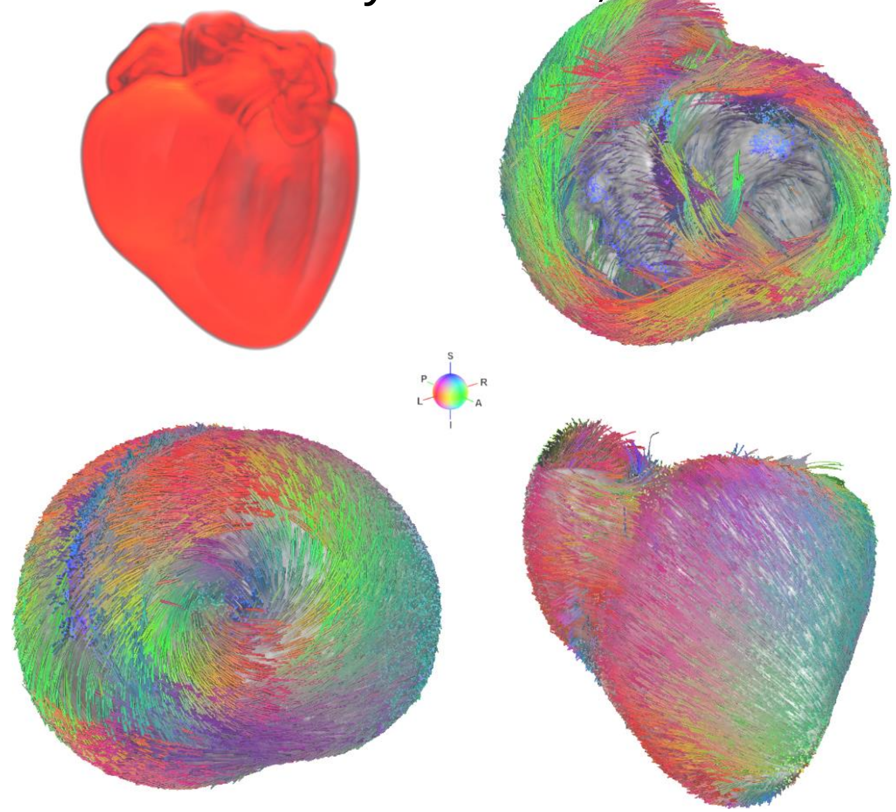
[J.M. Peyrat, et al., MICCAI'06, TMI 26(11), 2007]

Database

- 7 canine hearts from JHU
- Anatomical MRI and DTI

- *Average cardiac structure*
- Variability of fibers, sheets





Freely available at **http://www-sop.inria.fr/asclepios/data/heart**

A Statistical Atlas of the Cardiac Fiber Structure

[J.M. Peyrat, et al., MICCAI'06, TMI 26(11), 2007]

Database

- 7 canine hearts from JHU
- Anatomical MRI and DTI

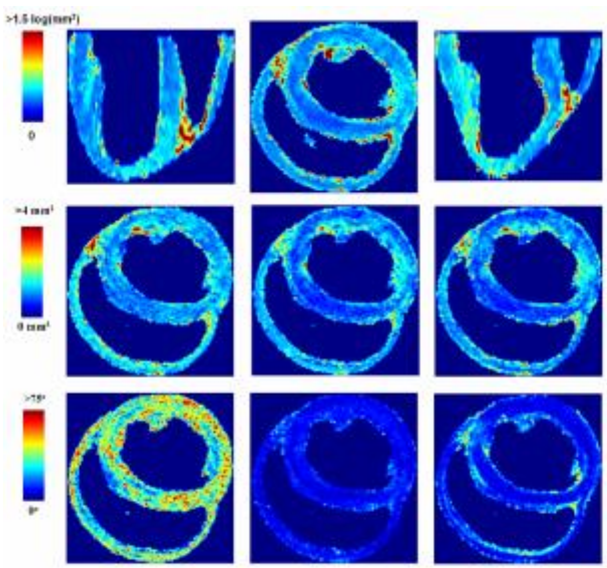
Method

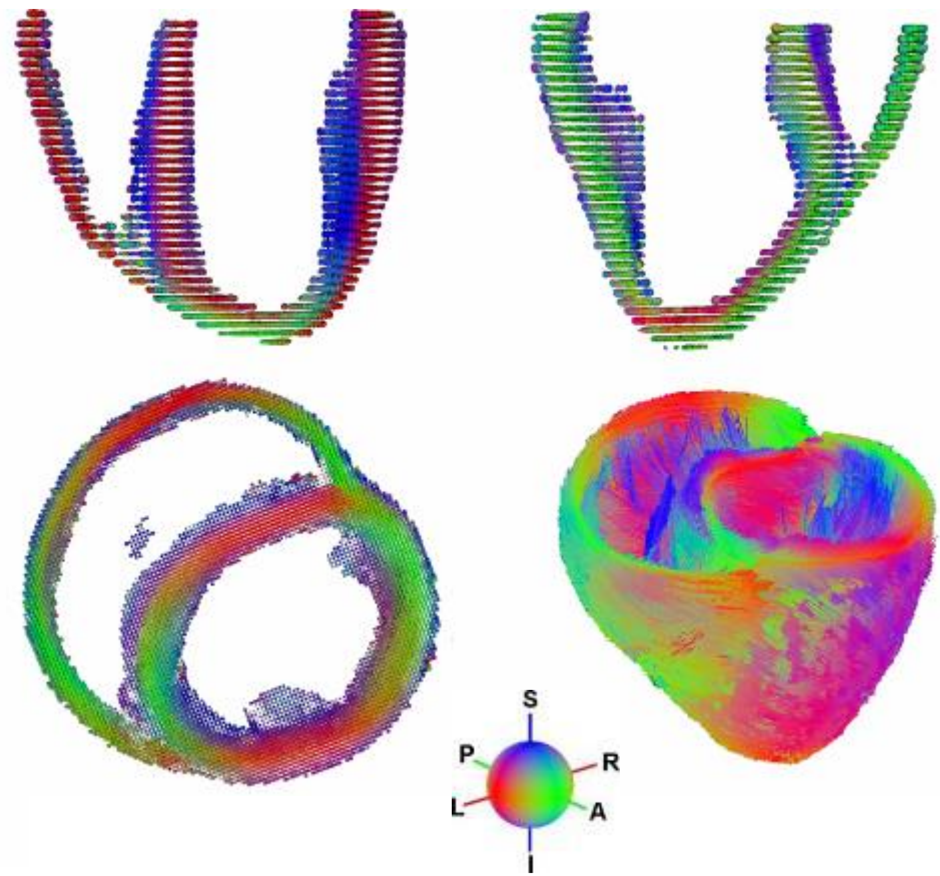
- Normalization based on aMRIs
- Log-Euclidean statistics of Tensors

Norm covariance

Eigenvalues covariance (1^{st}, 2^{nd}, 3^{rd})

Eigenvectors orientation covariance (around 1st, 2nd, 3rd)

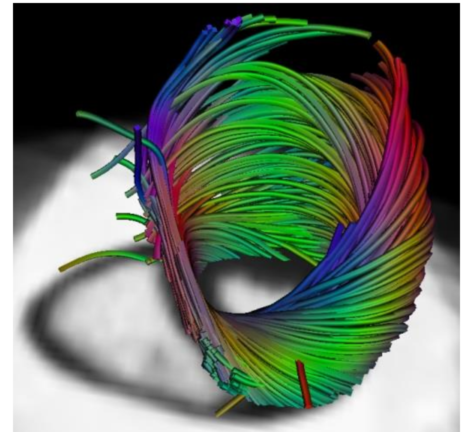




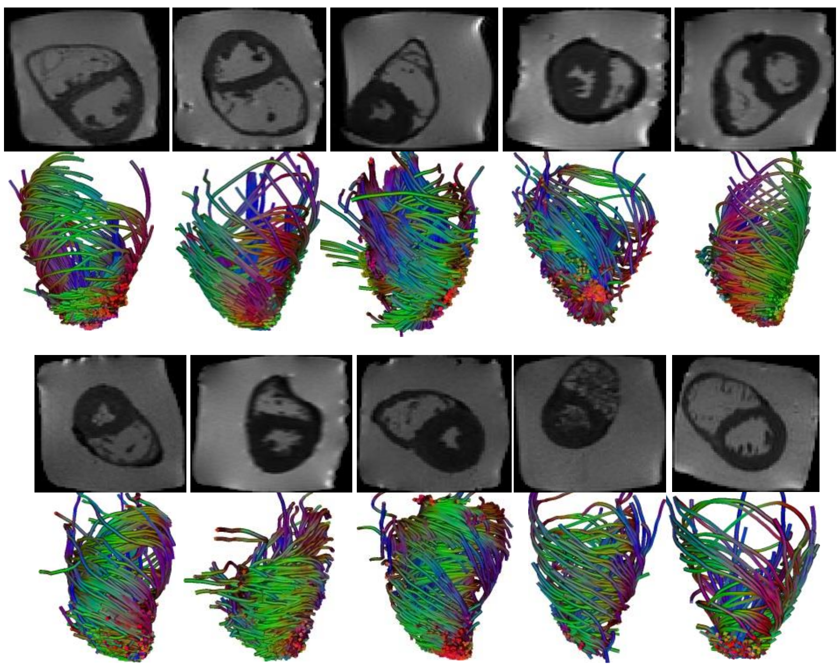
Diffusion model of the human heart

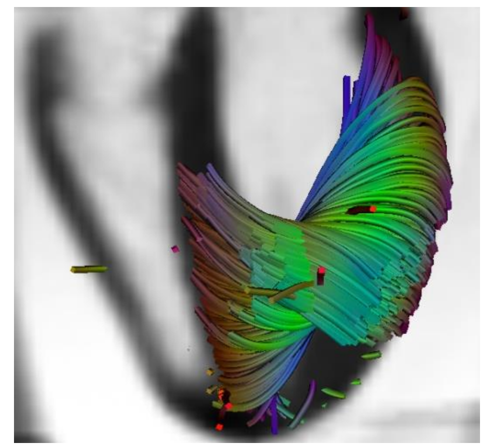
10 human ex vivo hearts (CREATIS-LRMN, Lyon, France)

- Classified as healthy (controlling weight, septal thickness, pathology examination)
- Acquired on 1.5T MR Avento Siemens
  - bipolar echo planar imaging, 4 repetitions, 12 gradients
- Volume size: 128×128×52, 2 mm resolution



Fiber tractography in the left ventricle

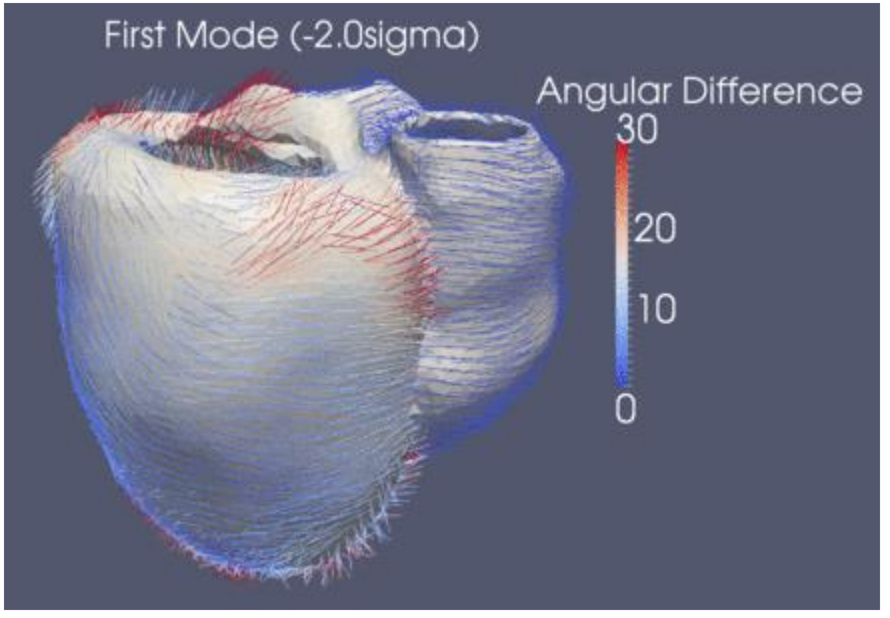


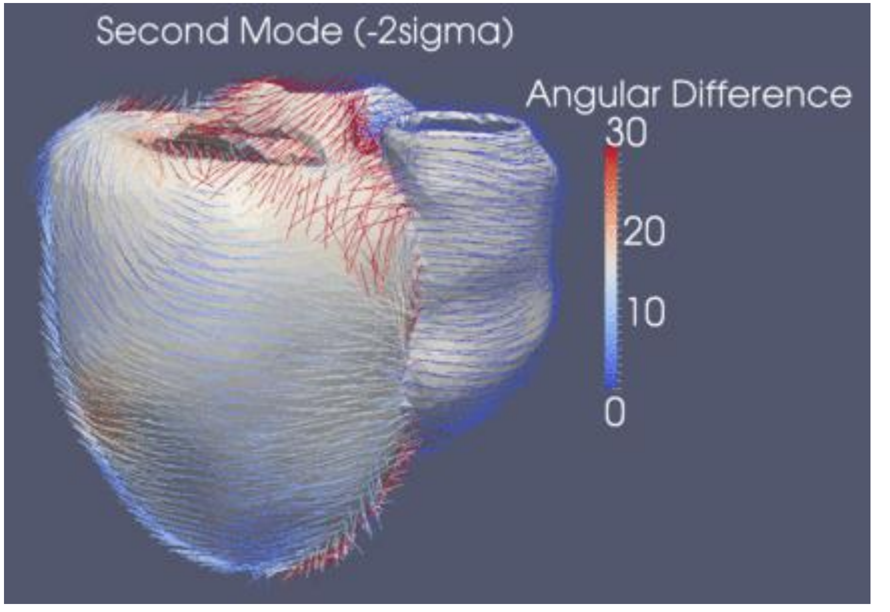


Helix angle highly correlated to the transmural distance

[H. Lombaert Statistical Analysis of the Human Cardiac Fiber Architecture from DT-MRI, ISMRM 2011, FIMH 2011]

A Statistical Atlas of the Cardiac Fiber Structure





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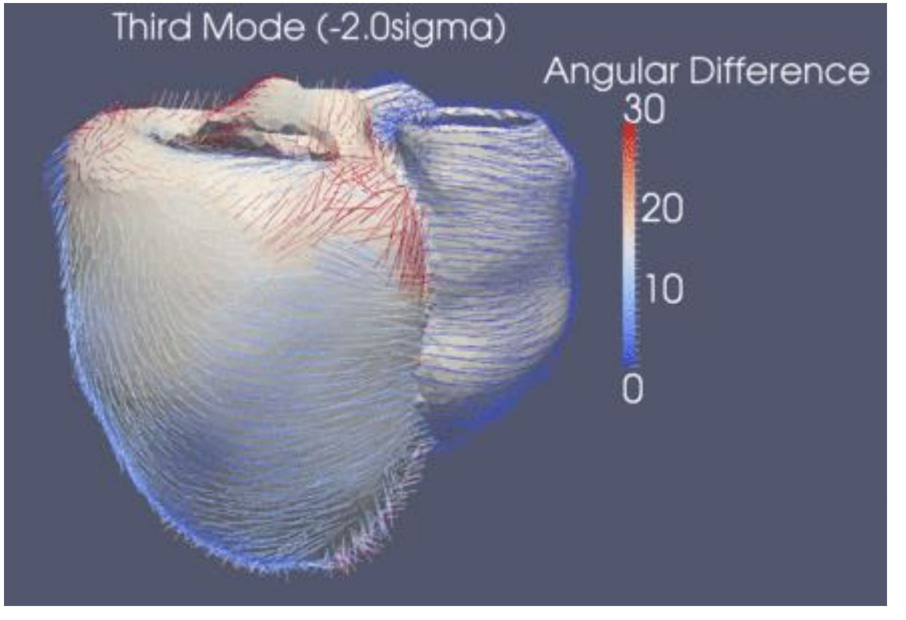
[R. Mollero, M.M Rohé, et al, to appear in FIMH 2015]



Conclusion

The Riemannian computing framework

- Integral or sum in M: minimize an intrinsic functional
  - Fréchet / Karcher mean
  - Filtering, convolution through weighted means
- The exponential chart is the basic tool
  - Gradient descent is geodesic walking
  - Intrinsic numerical scheme for Laplace Beltrami

Extrapolation of sparse tensor measurements

- Easy with log-Euclidean metrics
- Directional discontinuities (faults) can be imposed
  - Anisotropic filtering with structure tensor