An introduction to large dimensional array processing

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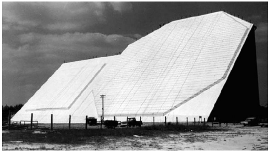
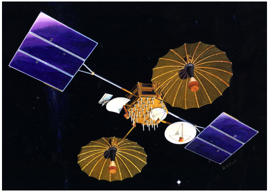
Ecole d'été de Peyresq 2017

Development of antenna arrays in the 20-th century

- **1905.** First known use of an array of antennas by Braun (Physics Nobel Prize), who discovers transmit *beamforming*.
- **1940.** Germany builds the first uniform circular array, called *Wullenweber*, for radio direction finding.
- **1960.** USA builds the active radar array ESAR (over 8000 elements).
- **1983.** 30-elements array used in the TDRSS satellite system.
- **1995.** Phased array embedded in combat aircrafts.



(a) ESAR



(b) TDRSS

(c) Aircraft radar

Antenna arrays and mobile communications (1)

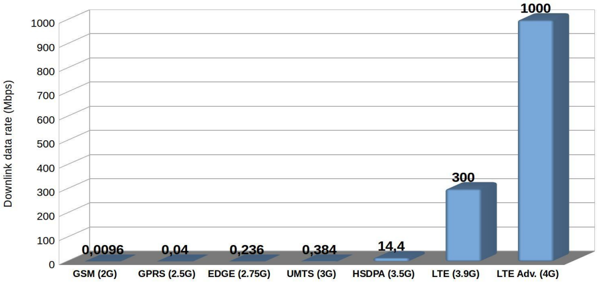


Figure : Evolution of downlink data rates (Mbps), from 2G to 4G

- TDMA, FDMA, CDMA, OFDMA.
- SDMA: No exploitation until LTE (MIMO 4x4) and LTE Adv. (MIMO 8x8).

Antenna arrays and mobile communications (2)

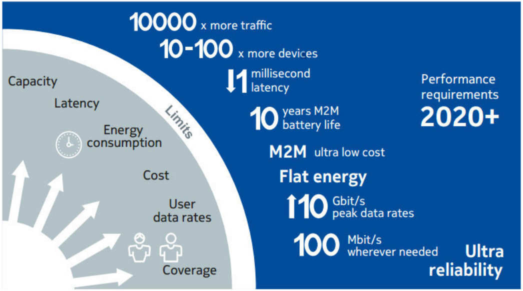
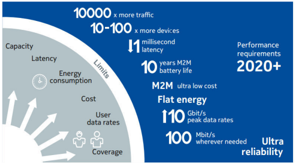


Figure : Requirements for future 2020 mobile standards (*source: Nokia*)

Antenna arrays and mobile communications (3)



Key features

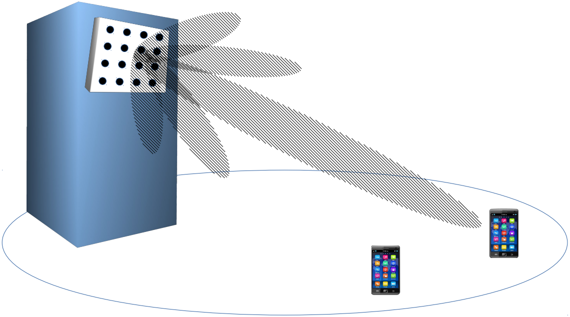
- Extreme densification of cells
- mmWave (30 GHz to 300 GHz)
- Massive MIMO (up to 120 antennas at base stations)

Challenges

- Green communications
- Co-user and co-channel interference
- Propagation of mmWaves

SDMA (1)

- **Scenario.**
  - BS equipped with a URA of $M \times M$ antennas,
  - K UTs equipped with a single antenna,
  - Line of sight between BS and UTs (single path model).



SDMA (2)

- **Model.** At discrete time n, the k-th UT receives the (baseband) signal,

$$y_n^{(k)} = \alpha_k \mathbf{b}(\theta_k, \phi_k)^* \mathbf{x}_n + v_n^{(k)},$$

  - $\alpha_k \in \mathbb{C}$ is a fading coefficient,
  - $\mathbf{x}_n \in \mathbb{C}^{M^2}$ in the BS transmit signal,
  - $v_n^{(k)}$ is an additive noise,
  - $\mathbf{b}(\theta_k, \phi_k) = \mathbf{a}(\theta_k) \otimes \mathbf{a}(\phi_k)$ represents the UT steering vector with

$$\mathbf{a}(u) = \Big(1, \exp(\mathrm{i}u), \ldots, \exp(\mathrm{i}(M-1)u)\Big)^T,$$

  and where θ_k, ϕ_k are two angles characterizing the direction of the UT.

SDMA (3)

- **Downlink beamforming.** Assuming $K \leq M^2$ and perfectly known directions $(\theta_1, \phi_1), \ldots, (\theta_K, \phi_K)$, the BS transmits

$$\mathbf{x}_n = \mathbf{B} \left(\mathbf{B}^* \mathbf{B} \right)^{-1} \mathbf{s}_n,$$

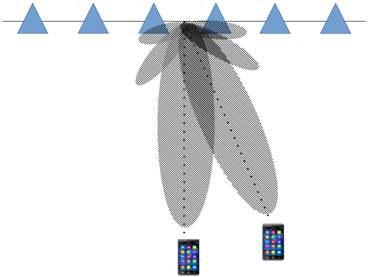
where

  - $\mathbf{s}_n = \left(s_n^{(1)}, \ldots, s_n^{(K)} \right)^T \in \mathbb{C}^K$ contains the K symbols sent to the UTs ;
  - $\mathbf{B} = [\mathbf{b}(\theta_1, \phi_1), \ldots, \mathbf{b}(\theta_K, \phi_K)]$.

Beamforming eliminates spatial interference between UTs, regardless the spacing between angles $(\theta_1, \phi_1), \ldots, (\theta_K, \phi_K)$:

$$y_n^{(k)} = \alpha_k s_n^{(k)} + v_n^{(k)}.$$

SDMA (4)



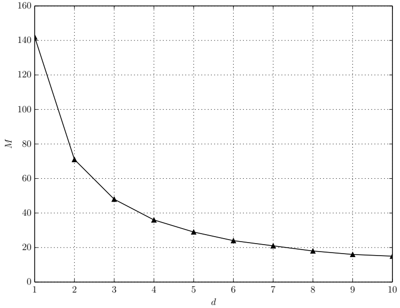


Figure : Minimal number of antennas M for DoA (azimuthal component) separation against UTs distance d in meters (uplink), for a standard beamformer and antennas spacing of half the wavelength **(distance UTs-BS = 100m)**

SDMA (5)

- **UTs separation.** Massive antenna arrays are needed to separate the DoA of closely spaced UT, with a spacing of the order of a beamwidth

$$\Delta\theta \approx \frac{2\pi}{M}.$$

- **Sample size.** To estimate closely spaced DoA, usual techniques require a large number N of samples, usually $N \gg M$, which may not be possible with future requirements.

- **"Spatial"cognitive radio.** Secondary BS must be able to perform detection on narrow angular sectors, with a limited number of observations.

SDMA (6)

- Limitations may essentially come from the uplink transmission, where accurate detection and DoA estimation, and reliable beamforming methods are needed to perform SDMA.

- **Beamforming with large arrays in other contexts.**
  - [Adhikary et al.'13] SDMA via conventional beamforming (using eigenvectors of the channel spatial correlation matrix)
  - [Sharif-Hassibi'05] SDMA via random beamforming and capacity analysis
  - [Alkhateeb et al.'15] Digital-analog hybrid beamforming

Statistical model and usual inference problems (1)

- **Scenario.**
  - ULA of M sensors
  - $K < M$ narrowband and far-field source signals with spatial frequencies $\theta_1, \ldots, \theta_K$
  - N observations $\mathbf{y}_1, \ldots, \mathbf{y}_N$

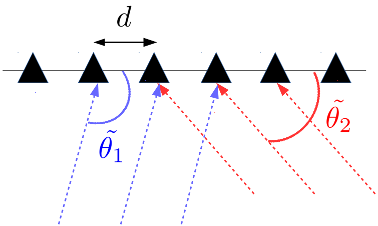


Figure : ULA with 2 sources at wavelength λ, with "physical" angle (DoA) $\tilde{\theta}_1, \tilde{\theta}_2$, and with corresponding "electrical" angle $\theta_k = 2\pi \frac{d}{\lambda} \cos\left(\tilde{\theta}_k\right)$

Statistical model and usual inference problems (2)

- **Received signal.**

$$\mathbf{y}_n = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_{k,n} + \mathbf{v}_n = \mathbf{A}\mathbf{s}_n + \mathbf{v}_n$$

  - **Steering vectors.** $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ and $\mathbf{a}(\theta) = (1, \mathrm{e}^{\mathrm{i}\theta}, \ldots, \mathrm{e}^{\mathrm{i}(M-1)\theta})^T$
  - **Source signals.** $\mathbf{s}_n = (s_{1,n}, \ldots, s_{K,n})^T$
  - **Additive noise.** $\mathbf{v}_n = (v_{1,n}, \ldots, v_{M,n})^T$

- **Statistical model.** For the remainder, we consider $\mathbf{s}_1, \ldots, \mathbf{s}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^K}(\mathbf{0}, \mathbf{\Gamma})$ and $\mathbf{v}_1, \ldots, \mathbf{v}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \sigma^2 \mathbf{I})$ which implies

$$\mathbf{y}_1, \ldots, \mathbf{y}_N \text{ i.i.d. } \mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R})$$

  where $\mathbf{R}$ is the spatial covariance matrix given by

$$\mathbf{R} = \mathbf{A}\mathbf{\Gamma}\mathbf{A}^* + \sigma^2 \mathbf{I}.$$

Statistical model and usual inference problems (3)

- **Detection.**
  - Test for the presence of one or more sources
  - Estimation of the source number K
- **DoA estimation.**
- **Beamforming.**
  - Estimation of the transmit signals $s_{1,n}, \ldots, s_{K,n}$
  - Estimation of the SINR

Statistical model and usual inference problems (4)

- **2nd order statistics.** All the information on K and $\theta_1, \ldots, \theta_K$ is contained in the eigenvalues and eigenvectors of $\mathbf{R}$.

- **Spectral decomposition.**

$$\mathbf{R} = \sum_{k=1}^{K} \lambda_k \mathbf{u}_k \mathbf{u}_k^* + \sigma^2 \underbrace{\sum_{k=K+1}^{M} \mathbf{u}_k \mathbf{u}_k^*}_{:=\mathbf{\Pi}}$$

  - $\lambda_1 \geq \ldots \geq \lambda_K > \lambda_{K+1} = \ldots = \lambda_M = \sigma^2$ are the eigenvalues
  - $\mathbf{u}_1, \ldots, \mathbf{u}_M$ are the associated orthonormal eigenvectors
  - $\mathbf{\Pi}$ is the orthogonal projection matrix onto the noise subspace

Statistical model and usual inference problems (5)

- **Detection and eigenvalues.**

$$K = \text{card}\left\{k : \lambda_k > \sigma^2\right\}$$

- **DoA and eigenvectors.** $\theta_1, \ldots, \theta_K$ are the unique zeros of the function

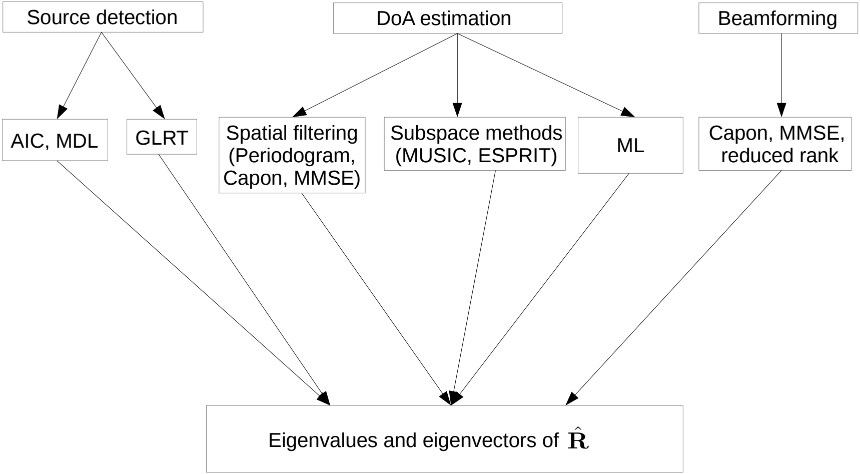
$$\theta \mapsto \|\mathbf{\Pi}\mathbf{a}(\theta)\|_2^2 = 1 - \sum_{k=1}^{K} \left|\mathbf{a}(\theta)^* \mathbf{u}_k\right|^2$$

$\mathbf{R}$ is not observable in practice and is usually replaced by the Sample Covariance Matrix (SCM)

$$\hat{\mathbf{R}}_N = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^*$$

which is a sufficient statistic.

Statistical model and usual inference problems (6)



Statistical model and usual inference problems (7)

- **Standard asymptotic regime.** For M, N fixed, the statistical performance of array processing methods is usually hard to predict, and the large sample size regime is considered:

$$M \text{ fixed}, \ N \to \infty$$

- **SCM.** Asymptotic performance results are mostly based on the fact that

$$\hat{\mathbf{R}}_N \xrightarrow[N\to\infty]{\text{a.s.}} \mathbf{R}$$

with Gaussian fluctuations.

- **Practical use.** Theoretical results in the large sample size can be used "safely" as long as

$$N \gg M.$$

Towards large dimensional array processing (1)

- **Large dimension paradigm.** If M is large and/or N is limited (short time duration/stationarity), N should be assumed to be of the same order of magnitude than M:

$$M \asymp N.$$

- **New asymptotic regime.** This situation is better described by the large dimensional regime

$$M, N \to \infty \text{ and } \frac{M}{N} \to c > 0.$$

Towards large dimensional array processing (2)

- **SINR.** When $M \to \infty$ and $\mathbb{E}[\mathbf{s}_n \mathbf{s}_n^*] = \mathbf{I}$, the SINR after beamforming is unbounded

$$\mathrm{SINR} = \frac{\|\mathbf{a}(\theta_k)\|^4 \, \mathbb{E}\,|s_{k,n}|^2}{\sum_{\ell \neq k} |\mathbf{a}(\theta_k)^* \mathbf{a}(\theta_\ell)|^2 \, \mathbb{E}\,|s_{\ell,n}|^2 + \|\mathbf{a}(\theta_k)\|^2 \sigma^2} = \frac{M}{\sigma^2} + \mathcal{O}(1)$$

- **Normalization.** To keep the SINR bounded, we consider the modified model

$$\mathbf{y}_n = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_{k,n} + \mathbf{v}_n,$$

where $\mathbf{a}(\theta) = \frac{1}{\sqrt{M}} \Big(1, \exp(\mathrm{i}\theta), \ldots, \exp(\mathrm{i}(M-1)\theta)\Big)^T$ is now unit norm.

  - The SINR after beamforming is $\mathcal{O}(1)$
  - The SINR per sensor is $\mathcal{O}\left(\frac{1}{M}\right)$

Towards large dimensional array processing (3)

- **Small number of sources.** $K \ll M$ (single path propagation, after spatial filtering ...)

$$K \text{ fixed while } M \to \infty$$

- **Large number of sources.** $K \asymp M$ (multipath propagation, clutter, ...)

$$K \to \infty \text{ such that } \frac{K}{M} \to d > 0.$$

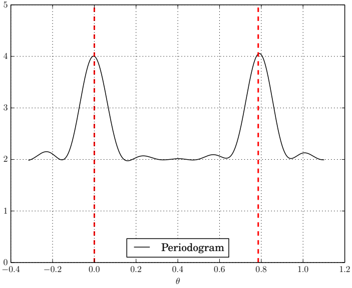
Towards large dimensional array processing (4)

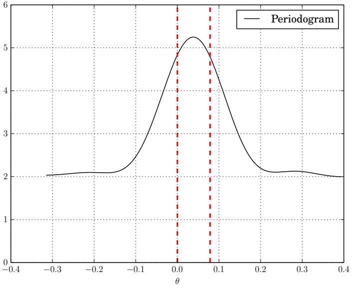
- **Widely spaced DoA.** $|\theta_k - \theta_\ell| \gg \frac{2\pi}{M}$

$$\theta_1, \ldots, \theta_K \text{ fixed as } M \to \infty$$

- **Closely spaced DoA.** $|\theta_k - \theta_\ell| \asymp \frac{2\pi}{M}$

$$\theta_k = \theta_\ell + \frac{\alpha}{M}, \; \alpha \text{ fixed as } M \to \infty$$





Towards large dimensional array processing (5)

- Behaviour of the SCM $\hat{\mathbf{R}}_N$, the sample eigenvalues and eigenvectors as $M, N \to \infty$?
- Performance of standard methods in the large dimensional regime vs large sample size regime ? Closely spaced DoA scenario ?
- New methods exploiting the behaviour of $\hat{\mathbf{R}}_N$? Theoretical performance ?

Summary of the main notations (1)

- M sensors, N samples, K sources, DoA $\theta_1, \ldots, \theta_K$

$$\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)] \text{ and } \mathbf{a}(\theta) = \frac{1}{\sqrt{M}} \left(1, \mathrm{e}^{\mathrm{i}\theta}, \ldots, \mathrm{e}^{\mathrm{i}(M-1)\theta}\right)^T.$$

- **Large sample size regime.** Denoted $N \to \infty$. $M, K, \theta_1, \ldots, \theta_K$ are fixed.
- **Large dimensional regime.** $M = M(N)$ is a function of N such that

$$c_N = \frac{M}{N} \xrightarrow[N\to\infty]{} c > 0.$$

This regime is denoted for clarity'sake $M, N \to \infty$. $K, \theta_1, \ldots, \theta_K$ may or may not depend on N, and we will add subscript N for all quantities depending on M, N.

Summary of the main notations (2)

- **Covariance matrix.**

$$\mathbf{R} = \mathbf{A}\boldsymbol{\Gamma}\mathbf{A}^* + \sigma^2\mathbf{I} = \sum_{k=1}^{K} \lambda_k \mathbf{u}_k \mathbf{u}_k^* + \sigma^2 \sum_{k=K+1}^{M} \mathbf{u}_k \mathbf{u}_k^*$$

where $\lambda_1 \geq \ldots \geq \lambda_K > \sigma^2$ (mult. $M-K$) are the eigenvalues associated with the orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_M$.

- **Sample covariance matrix (SCM).**

$$\hat{\mathbf{R}}_N = \frac{1}{N}\sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^* = \sum_{k=1}^{M} \hat{\lambda}_{k,N} \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^*$$

where $\hat{\lambda}_{1,N} \geq \ldots \geq \hat{\lambda}_{M,N} \geq 0$ are the eigenvalues associated with the orthonormal eigenvectors $\hat{\mathbf{u}}_{1,N}, \ldots, \hat{\mathbf{u}}_{M,N}$.

- **Projections.** $\boldsymbol{\Pi} = \sum_{k=K+1}^{M} \mathbf{u}_k \mathbf{u}_k^*$ and $\hat{\boldsymbol{\Pi}}_N = \sum_{k=K+1}^{M} \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^*$.

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Single source detection

- **Formulation.** The detection of a single source is usually formulated through a hypothesis test, by "forgetting" the array manifold parametrization:

$$\mathcal{H}_0 : \mathbf{y}_n = \mathbf{v}_n \;\; \sim \mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right) \qquad \text{(pure noise)}$$
$$\mathcal{H}_1 : \mathbf{y}_n = \mathbf{h}\mathbf{s}_n + \mathbf{v}_n \;\; \sim \mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \mathbf{h}\mathbf{h}^* + \sigma^2 \mathbf{I}\right) \qquad \text{(one source)}$$

where $\mathbf{h} \in \mathbb{C}^M \backslash \{\mathbf{0}\}$ is a deterministic unknown vector.

GLRT

The GLRT is equivalent to compute the test

$$\hat{T}_N = \frac{\hat{\lambda}_{1,N}}{\frac{1}{M}\mathrm{tr}\hat{\mathbf{R}}_N} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \epsilon$$

where the threshold ϵ is set according to a desired false alarm probability.

False alarm probability

- **Finite M, N.** Under $\mathcal{H}_0$, expression of the exact distribution of $\hat{T}_N$ is well-known [Schuurmann et al. '73].
  - Untractable expression and computationaly expensive even for moderate M
  - No insight on the fluctuations of $\hat{T}_N$
- **Large sample size.** Under $\mathcal{H}_0$, from the LLN,

$$\hat{T}_N \xrightarrow[N\to\infty]{a.s.} 1.$$

No simple expression of the asymptotic distribution of $\hat{\lambda}_{1,N}$ (under convenient renormalization) is known in the regime $N \to \infty$.

Large dimensional regime - Marcenko-Pastur distribution (1)

- Considering the joint distribution of $\hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{M,N}$ is not relevant any regime where $M \to \infty$. Instead, we focus on the proportion of sample eigenvalues inside a Borel set $A \subset \mathbb{R}$:

$$\hat{\mu}_N(A) = \frac{1}{M} \text{card} \left\{ m : \hat{\lambda}_{m,N} \in A \right\}$$

- **Empirical spectral distribution.**

$$\hat{\mu}_N = \frac{1}{M} \sum_{m=1}^{M} \delta_{\hat{\lambda}_{m,N}}$$

where δ_x is the Dirac measure at point x.

Random probability measure representing the histogram of the sample eigenvalues

Large dimensional regime - Marcenko-Pastur distribution (2)

Theorem [Marcenko-Pastur'67]

If $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right)$, then with probability one,

$$\hat{\mu}_N \xrightarrow[M,N\to\infty]{w} \mu_{\sigma^2,c}$$

where $\mu_{\sigma^2,c}$ is a deterministic probability measure given by

$$\mathrm{d}\mu_{\sigma^2,c}(\lambda) = \left(1 - \frac{1}{c}\right)^+ \delta_0(\mathrm{d}\lambda) + \frac{\sqrt{\left(\lambda - \lambda^-\right)\left(\lambda^+ - \lambda\right)}}{2\pi\sigma^2 c\lambda} \mathbb{1}_{[\lambda^-,\lambda^+]}(\lambda)\mathrm{d}\lambda.$$

and $\lambda^- = \sigma^2(1-\sqrt{c})^2$, $\lambda^+ = \sigma^2(1+\sqrt{c})^2$.

Large dimensional regime - Marcenko-Pastur distribution (3)

- **Corollary.** $\hat{\mathbf{R}}_N$ is no more a consistent estimator of $\mathbf{R}_N$, i.e.

$$\left\|\hat{\mathbf{R}}_N - \mathbf{R}_N\right\|_2 \quad \underset{M,N}{\overset{a.s.}{\not\longrightarrow}} \quad 0.$$

- **Histogram.** For all $\varphi \in \mathcal{C}_b(\mathbb{R})$, with probability one as $M, N \to \infty$,

$$\frac{1}{M}\sum_{m=1}^{M} \varphi(\hat{\lambda}_{m,N}) =$$

$$\left(1 - \frac{N}{M}\right)^{+} \varphi(0) + \frac{1}{2\pi}\int_{\lambda_N^-}^{\lambda_N^+} \varphi(\lambda) \frac{\sqrt{\left(\lambda - \lambda_N^-\right)\left(\lambda_N^+ - \lambda\right)}}{\lambda\sigma^2 M/N} \mathrm{d}\lambda + o(1)$$

with $\lambda_N^{\pm} = \sigma^2 \left(1 \pm \sqrt{M/N}\right)^2$.

Large dimensional regime - Marcenko-Pastur distribution (4)

- **Universality.** The Marcenko-Pastur theorem also holds in the non-Gaussian case, still assuming that $\mathbb{E}[\mathbf{y}_1] = \mathbf{0}$ and $\mathbb{E}[\mathbf{y}_1\mathbf{y}_1^*] = \sigma^2\mathbf{I}$. [Yin'86]

- **Spectral statistics.** For all φ analytic on a neighborhood of $[\lambda_M^-, \lambda_M^+]$,

$$N\left(\frac{1}{M}\sum_{m=1}^{M}\varphi(\hat{\lambda}_{m,N}) - \int_{\mathbb{R}}\varphi(\lambda)\mathrm{d}\mu_{\sigma^2,c_N}(\lambda)\right) \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,\gamma^2).$$

$\Rightarrow$ Fast convergence [Bai-Silverstein'04]

$$\varphi(z) = z^\ell$$
$$\varphi(z) = \log(z)$$
$$\vdots$$

Large dimensional regime - Marcenko-Pastur distribution (5)

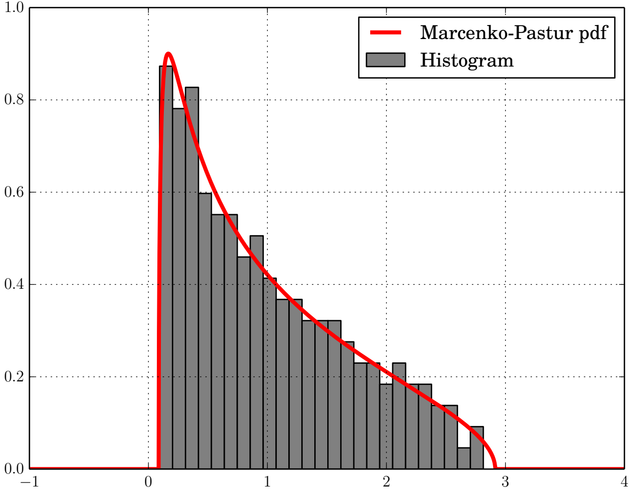


Figure : Marcenko-Pastur distribution and histogram of the sample eigenvalues for $M = 200,\ N = 400,\ \sigma^2 = 1$

Large dimensional regime - Marcenko-Pastur distribution (6)

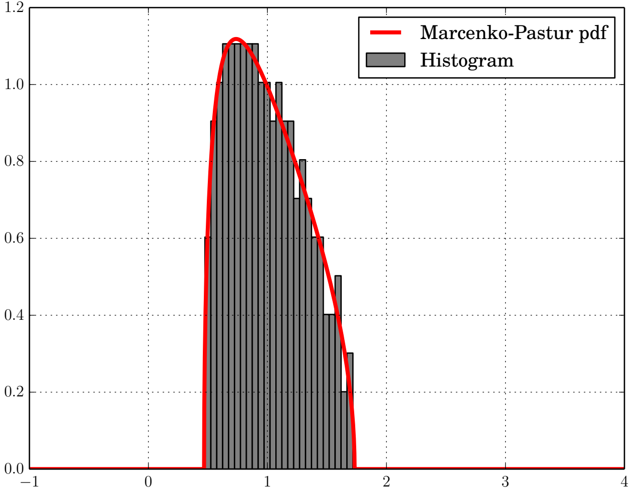


Figure : Marcenko-Pastur distribution and histogram of the sample eigenvalues for $M = 200,\ N = 2000,\ \sigma^2 = 1$

Large dimensional regime - Marcenko-Pastur distribution (7)

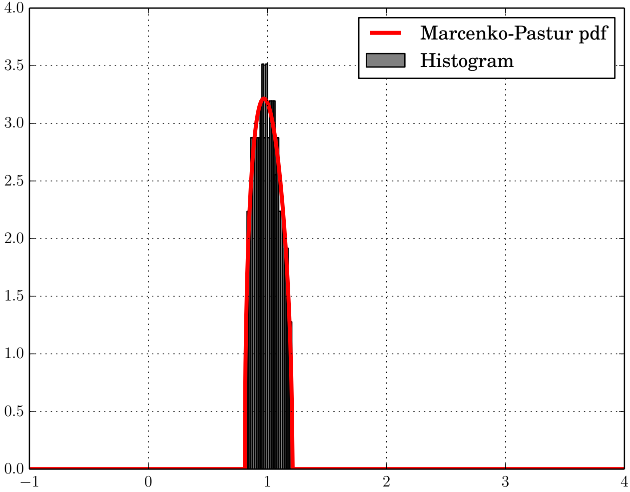


Figure : Marcenko-Pastur distribution and histogram of the sample eigenvalues for $M = 200,\ N = 20000,\ \sigma^2 = 1$

Large dimensional regime - Marcenko-Pastur distribution (8)

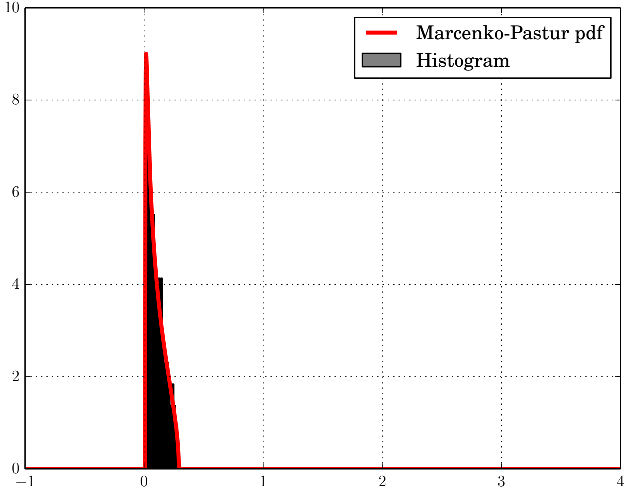
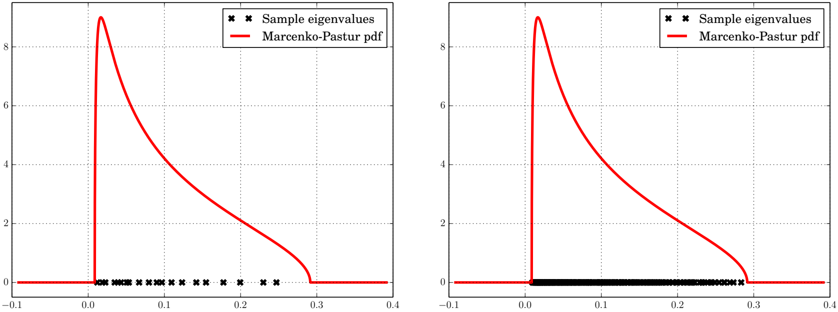


Figure : Marcenko-Pastur distribution and histogram of the sample eigenvalues for $M = 200,\ N = 400,\ \sigma^2 = 0.1$

Large dimensional regime - Extreme eigenvalues (1)



(a) $M = 20, N = 40$ (b) $M = 200, N = 400$

Figure : Location of the sample eigenvalues w.r.t. the MP distribution

Large dimensional regime - Extreme eigenvalues (2)

Theorem [Yin-Bai-Krishnaiah'88, Bai-Yin'93]

Under the assumptions of the Marcenko-Pastur theorem and if $c \leq 1$,

$$\hat{\lambda}_{1,N} \xrightarrow[M,N\to\infty]{a.s.} \sigma^2 \left(1+\sqrt{c}\right)^2$$

$$\text{and } \hat{\lambda}_{M,N} \xrightarrow[M,N\to\infty]{a.s.} \sigma^2 \left(1-\sqrt{c}\right)^2.$$

- **Corollary.** For any $\epsilon > 0$, all the sample eigenvalues concentrate inside

$$\left(\sigma^2\left(1-\sqrt{\frac{M}{N}}\right)^2-\epsilon, \sigma^2\left(1+\sqrt{\frac{M}{N}}\right)^2+\epsilon\right)$$

  w.p.1 for all large M, N.

- **Universality.** The result holds in the non-Gaussian case under the finite fourth moment assumption $\mathbb{E}|\mathbf{y}_{1,1}|^4 < \infty$.

Large dimensional regime - Extreme eigenvalues (3)

Theorem [Johnstone'01]

Under the assumptions of the Marcenko-Pastur theorem,

$$N^{2/3}\frac{\hat{\lambda}_{1,N}-\sigma^2\left(1+\sqrt{c_N}\right)^2}{\sigma^2\left(1+\sqrt{c_N}\right)\left(1+\frac{1}{\sqrt{c_N}}\right)^{1/3}}\xrightarrow[M,N\to\infty]{\mathcal{D}}\mathrm{TW}(2)$$

- **Tracy-Widom distribution.** TW(2) is the 2nd Tracy-Widom distribution [Tracy-Widom'96] with cdf

$$F(x)=\exp\left(-\int_x^\infty (t-x)q(t)^2\mathrm{d}t\right),$$

where q solves the Painlevé II differential equation $q^{(2)}(t)=tq(t)+2q(t)^3$ with some boundary condition.

Large dimensional regime - Extreme eigenvalues (4)

- **Fluctuations.** The fluctuations of $\hat{\lambda}_{1,N}$ around its limiting value are smaller than the "usual" $N^{-1/2}$ rate:

$$\hat{\lambda}_{1,N} = \sigma^2 \left(1 + \sqrt{\frac{M}{N}}\right)^2 + \mathcal{O}_{\mathbb{P}}\left(\frac{1}{N^{2/3}}\right)$$

- **Extensions.** A similar result holds for the smallest sample eigenvalue $\hat{\lambda}_{M,M}$. The Tracy-Widom also holds for certain non-Gaussian distributions.

Large dimensional regime - Extreme eigenvalues (5)

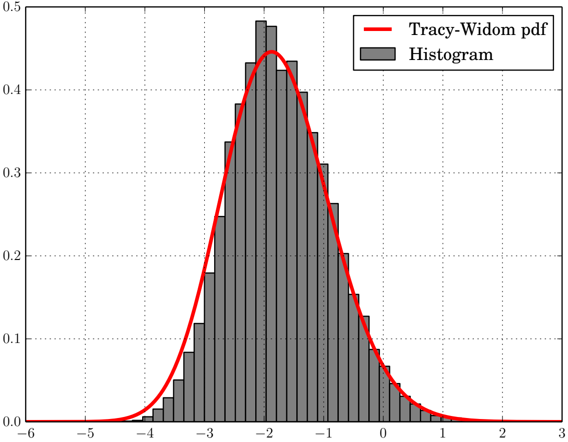


Figure : Tracy-Widom distribution and histogram of $\hat{\lambda}_{1,N}$, for $M = 20, N = 40$ and 20000 realizations

False alarm probability - Conclusion (1)

- **Fluctuations.** The denominator in $\hat{T}_N$ satisfies

$$\frac{1}{M}\mathrm{tr}\hat{\mathbf{R}}_N = \sigma^2 + \mathcal{O}_{\mathbb{P}}\left(\frac{1}{N}\right).$$

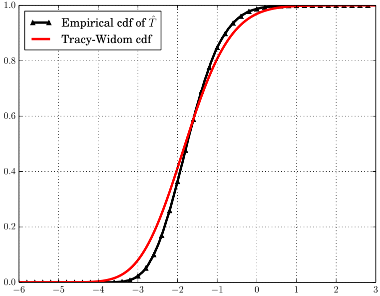
and its fluctuations are smaller than $\hat{\lambda}_{1,N}$.

Asymptotic False Alarm Probability

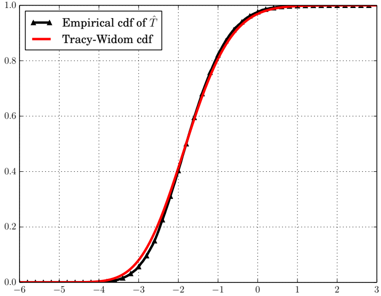
Under $\mathcal{H}_0$ and the conditions of Johnstone's theorem,

$$N^{2/3}\frac{\hat{T}_N - \left(1+\sqrt{c_N}\right)^2}{\left(1+\sqrt{c_N}\right)\left(1+\frac{1}{\sqrt{c_N}}\right)^{1/3}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathrm{TW}(2).$$

False alarm probability - Conclusion (2)



(a) $M = 20, N = 40$



(b) $M = 100, N = 200$

Figure : Empirical cdf of $\hat{T}_N$ under $\mathcal{H}_0$ (recentered and rescaled) and TW cdf

- In [Nadler'11], a correction to the TW distribution is proposed to improve the PFA approximation for moderate M, N.

Detection probability (1)

- **Finite M, N.** Under $\mathcal{H}_1$, no expression seems available in the literature.
- **LSS regime - 1st order.** From the LLN,

$$\hat{\lambda}_{1,N} \xrightarrow[N\to\infty]{a.s.} \sigma^2(1+\rho)$$

$$\sum_{k=2}^{M} \hat{\lambda}_{k,N} \xrightarrow[N\to\infty]{a.s.} (M-1)\sigma^2,$$

and thus

$$\hat{T}_N \xrightarrow[N\to\infty]{a.s.} \frac{1+\rho}{1+\rho/M}$$

where $\rho = \frac{\|\mathbf{h}\|^2}{\sigma^2}$ represents the SNR.

Detection probability (2)

- **LSS regime - 2nd order.** On the other hand, a straightforward application of the CLT leads to

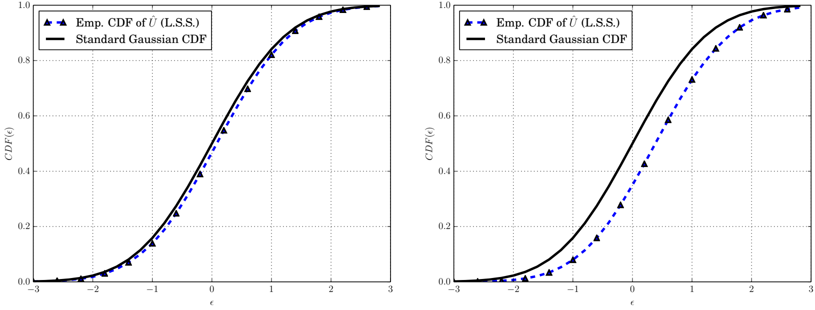
$$\sqrt{N}\left(\hat{\lambda}_{1,N} - \lambda_1\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma^4(1+\rho)^2\right)$$

$$\sqrt{N}\sum_{k=2}^{M}\left(\hat{\lambda}_{k,N} - \lambda_k\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma^4(M-1)\right)$$

from which we deduce

$$\sqrt{N}\left(\hat{T}_N - \frac{1+\rho}{1+\rho/M}\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \frac{(1-1/M)\,(1+\rho)^2}{(1+\rho/M)^4}\right).$$

Detection probability (3)



(a) $M = 10, N = 1000$ (b) $M = 40, N = 500$

Figure : Empirical cdf of $\hat{U} = \sqrt{N}\frac{(1+\rho/M)^2}{\sqrt{1-1/M(1+\rho)}}\left(\hat{T} - \frac{1+\rho}{1+\rho/M}\right)$ and $\mathcal{N}(0,1)$ cdf ($\rho = 5$)

Large dimensional regime - Escape from the bulk (1)

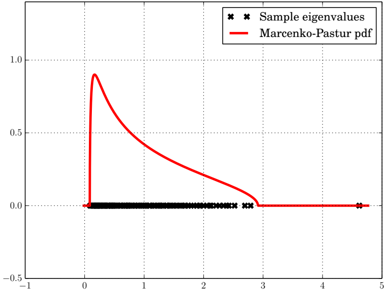
- Let $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R}_N)$, with $\mathbf{R}_N$ having eigenvalues $\lambda_{1,N} \geq \ldots \geq \lambda_{K,N} > \sigma^2$ (mult. $M - K$) s.t $\limsup \lambda_{1,N} < \infty$.
- When K is fixed with respect to M, $\mathbf{R}_N$ is a fixed rank perturbation of $\sigma^2 \mathbf{I}$ (Spiked Models).
- In that case, it holds (again)

$$\hat{\mu}_N = \frac{1}{M} \sum_{k=1}^{M} \delta_{\hat{\lambda}_{k,N}} \xrightarrow[M,N\to\infty]{w} \mu_{\sigma^2,c} \quad \text{a.s.},$$

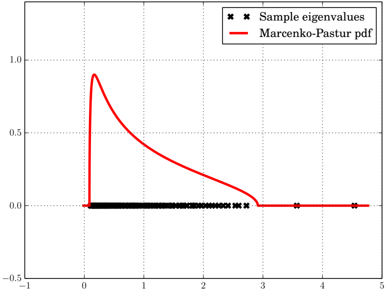
where μ is the Marcenko-Pastur distribution.

What about the individual behaviour of the sample eigenvalues $\hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{K,N}$?

Large dimensional regime - Escape from the bulk (2)



(a) $\lambda_1 = 4, \lambda_2 = 1.5$



(b) $\lambda_1 = 4, \lambda_2 = 3$

Figure : Phase transition in the spectrum of $\hat{\mathbf{R}}_N$ under $\mathcal{H}_1$ ($M = 100,\ N = 200,\ \sigma^2 = 1$)

Large dimensional regime - Escape from the bulk (3)

Theorem [Baik-Silverstein'06]

Let $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R}_N)$, with $\mathbf{R}_N$ having eigenvalues $\lambda_{1,N} \geq \ldots \geq \lambda_{K,N} > \sigma^2$ (mult. $M-K$) such that K si fixed w.r.t. N.

Then, for $k \in \{1, \ldots, K\}$,

- If $\lambda_{k,N} \xrightarrow[M,N\to\infty]{} \lambda_k > \sigma^2 (1+\sqrt{c})$,

$$\hat{\lambda}_{k,N} \xrightarrow[M,N\to\infty]{a.s.} \lambda_k + \frac{\sigma^2 c \lambda_k}{\lambda_k - \sigma^2}.$$

- If $\lambda_{k,N} \xrightarrow[M,N\to\infty]{} \lambda_k \leq \sigma^2 (1+\sqrt{c})$,

$$\hat{\lambda}_{k,N} \xrightarrow[M,N\to\infty]{a.s.} \sigma^2 \left(1+\sqrt{c}\right)^2.$$

Moreover, $\hat{\lambda}_{K+1,N} \to \sigma^2 (1+\sqrt{c})^2 \to 0$ a.s.

Large dimensional regime - Escape from the bulk (4)

- **Remark 1.** In particular, w.p.1 as $M, N \to \infty$,

$$\hat{\lambda}_{1,N} = \lambda_{1,N} + \frac{M}{N} \frac{\sigma^2 \lambda_{1,N}}{\lambda_{1,N} - \sigma^2} + o(1),$$

- **Remark 2.** The function

$$\phi_{\sigma^2,c}(\lambda) = \lambda + \frac{\lambda \sigma^2 c}{\lambda - \sigma^2}$$

is a one-to-one increasing mapping from $\left[\sigma^2 \left(1+\sqrt{c}\right), +\infty\right)$ to $\left[\sigma^2 \left(1+\sqrt{c}\right)^2, +\infty\right)$. It relates the spectra of $\mathbf{R}_N$ and $\hat{\mathbf{R}}_N$. In particular,

$$\phi_{\sigma^2,c}\left(\sigma^2\left(1+\sqrt{c}\right)\right) = \sigma^2\left(1+\sqrt{c}\right)^2.$$

Large dimensional regime - Escape from the bulk (5)

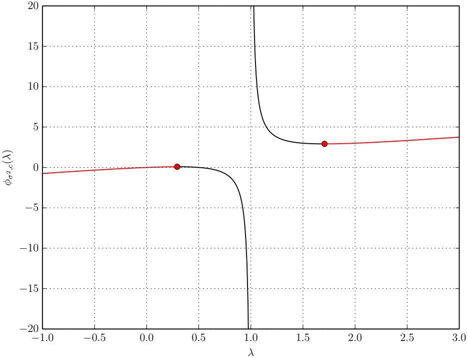


Figure : Plot of function $\lambda \mapsto \phi_{c,\sigma^2}(\lambda)$, with red points indicating couples $\left(\sigma^2(1 \pm \sqrt{c}), \sigma^2(1 \pm \sqrt{c})^2\right)$, with $\sigma^2 = 1$, $c = 0.5$

Large dimensional regime - Escape from the bulk (6)

- **Critical value.** If $\lambda_k \leq \sigma^2(1+\sqrt{c})$, the corresponding $\hat{\lambda}_{k,N}$ is asymptotically absorbed in the support of the M-P distribution. Otherwise, it escapes.
- **Extension.** The results still holds in the non-Gaussian case under the assumption that

$$\mathbf{y}_k = \mathbf{R}_N^{1/2}\mathbf{w}_k,$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_N$ are i.i.d. zero mean, with $\mathbb{E}|w_{1,1}|^2 = 1$ and $\mathbb{E}|w_{1,1}|^4 < \infty$.

Large dimensional regime - Escape from the bulk (7)

Theorem [Baik et al.'05]

Under the assumptions of the previous theorem, and if

$$\lambda_1 > \ldots > \lambda_K > \sigma^2(1+\sqrt{c}),$$

then

$$\sqrt{N}\frac{\hat{\lambda}_{k,N} - \phi_{\sigma^2,c_N}\left(\lambda_{k,N}\right)}{\sqrt{\lambda_{k,N}^2 - \frac{\lambda_{k,N}^2\sigma^4 c_N}{(\lambda_{k,N}-\sigma^2)^2}}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}\left(0,1\right).$$

Additionally, $\hat{\lambda}_{1,N},\ldots,\hat{\lambda}_{K,N}$ and the vector $\left(\hat{\lambda}_{K+1,N},\ldots,\hat{\lambda}_{M,N}\right)$ are asymptotically mutually independent.

Large dimensional regime - Escape from the bulk (8)

- **Fluctuations.** In particular, as $M, N \to \infty$,

$$\hat{\lambda}_{1,N} = \lambda_{1,N} + \frac{M}{N} \frac{\sigma^2 \lambda_{1,N}}{\lambda_{1,N} - \sigma^2} + \mathcal{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{N}}\right),$$

with an asymptotic variance given by

$$\xi_N^2 = \lambda_{k,N}^2 - \frac{M}{N} \frac{\lambda_{k,N}^2 \sigma^4}{\left(\lambda_{k,N} - \sigma^2\right)^2}.$$

- **Remark 1.** If $c_N \approx 0$, then $\xi_N^2 \approx \lambda_{k,N}^2$ $\Rightarrow$ large sample size regime.
- **Remark 2.** If $\lambda_{k,N} \approx \sigma^2(1 + \sqrt{c_N})$, then $\xi_N^2 \approx 0$ $\Rightarrow$ different fluctuations (Tracy-Widom).

Detection probability - Conclusion (1)

- **Detectability threshold.** If $\lambda_1 > \sigma^2(1+\sqrt{c})$, that is

$$\sqrt{c} < \lim_{M,N\to\infty} \frac{\|\mathbf{h}\|^2}{\sigma^2} < \infty,$$

  the $\hat{\lambda}_{1,N}$ escapes from the support of the M-P distribution.

- In the large dimensional regime, if for M, N large enough, the SNR ρ_N satisfies

$$\rho_N = \frac{\|\mathbf{h}\|^2}{\sigma^2} > \sqrt{\frac{M}{N}} + \epsilon,$$

  for a fixed $\epsilon > 0$, then the source is detectable.

- **Fluctuations.** Under $\mathcal{H}_1$, the denominator in $\hat{T}_N$ satisfies

$$\frac{1}{M}\mathrm{tr}\hat{\mathbf{R}}_N = \sigma^2 + \mathcal{O}_{\mathbb{P}}\left(\frac{1}{N}\right).$$

Detection probability - Conclusion (2)

Asymptotic detection probability

Under $\mathcal{H}_1$, if $\lim_{M,N\to\infty} \frac{\rho_N}{\sqrt{c_N}} > 1$,

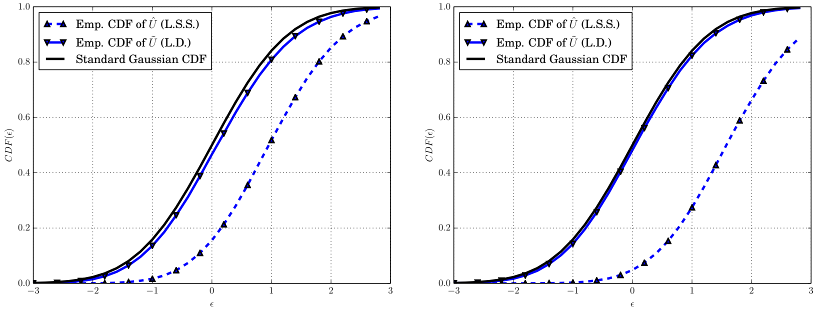
$$\sqrt{N}\frac{\hat{T}_N - \alpha_N}{\xi_N} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

where

$$\alpha_N = \frac{(1+\rho_N)\left(1+\frac{c_N}{\rho_N}\right)}{1-\frac{1}{M}+\frac{1}{M}(1+\rho_N)\left(1+\frac{c_N}{\rho_N}\right)} = (1+\rho_N)\left(1+\frac{c_N}{\rho_N}\right) + \mathcal{O}\left(\frac{1}{N}\right).$$

$$\xi_N^2 = \frac{(1+\rho_N)^2\left(1-\frac{c_N}{\rho_N^2}\right)}{\left(\sqrt{\frac{M-1}{M}}+\frac{(1+\rho_N)\left(1+\frac{c_N}{\rho_N}\right)}{\sqrt{M(M-1)}}\right)^4} = (1+\rho_N)^2\left(1-\frac{c_N}{\rho_N^2}\right) + \mathcal{O}\left(\frac{1}{N}\right).$$

Detection probability - Conclusion (3)



(a) $M = 40, N = 80$ (b) $M = 120, N = 240$

Figure : Empirical cdf of $\hat{U}$ and $\tilde{U} = \sqrt{N}\frac{\hat{T}_N - \alpha_N}{\xi_N}$ and $\mathcal{N}(0,1)$ cdf ($\rho_N = 5$)

Detection probability - Conclusion (4)

- **Correction.** By assuming K may increase with M, N, and keeping the terms $\mathcal{O}\left(\frac{K}{N}\right)$, we can prove that [Mestre'08]

$$\hat{\lambda}_{1,N} = \sigma^2 (1+\rho_N) \left(1 + \left(1 - \frac{1}{M}\right) \frac{c_N}{\rho_N}\right) + o(1),$$

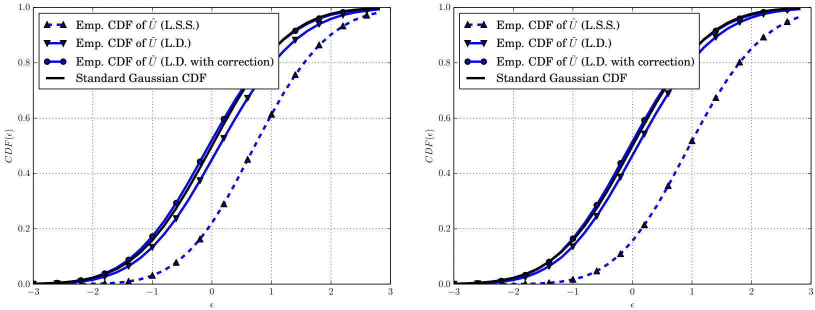
$$\frac{1}{M-1} \sum_{k=2}^{M} \hat{\lambda}_{k,N} = \sigma^2 \left(1 - \frac{c_N (1+\rho_N)}{M \rho_N}\right) + o(1).$$

w.p.1 for all large M, N, which gives the following correction for the asymptotic mean α_N (see Section 3 below):

$$\alpha_N = \frac{(1+\rho_N)\left(1 + \left(1 - \frac{1}{M}\right) \frac{c_N}{\rho_N}\right)}{\left(1 - \frac{1}{M}\right)\left(1 - \frac{c_N(1+\rho_N)}{M\rho_N}\right) + \frac{1}{M}(1+\rho_N)\left(1 + \frac{c_N}{\rho_N}\right)}.$$

A similar correction can be obtained for the asymptotic variance ξ_N^2.

Detection probability - Conclusion (5)



(a) $M = 20, N = 40$ (b) $M = 40, N = 80$

Figure : Empirical cdf of $\hat{U}$ and $\tilde{U} = \sqrt{N}\frac{\hat{T}_N - \alpha_N}{\xi_N}$ with and without correction, and $\mathcal{N}(0,1)$ cdf ($\rho_N = 5$)

Detection probability - Conclusion (6)

- **Exponential rate.** [Bianchi et al.'11] obtained a Large Deviations Principle for $\hat{T}_N$ under $\mathcal{H}_1$, in the large dimensional regime.
- **Other works.**
  - [Nadler'10] Analysis of AIC/MDL for source number estimation
  - [Kritchman-Nadler'11] Multiple hypothesis test for source detection

Contents

The MUSIC method (1)

- **Model.** $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. where

$$\mathbf{y}_n = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_{k,n} + \mathbf{v}_n = \mathbf{A}\mathbf{s}_n + \mathbf{v}_n \sim \mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \mathbf{R}\right),$$

  with $\mathbf{R} = \mathbf{A}\boldsymbol{\Gamma}\mathbf{A}^* + \sigma^2 \mathbf{I}$

- **Subspace method.** $\mathrm{span}\left\{\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)\right\} = \mathrm{span}\left\{\mathbf{u}_1, \ldots, \mathbf{u}_K\right\}$
- **Pseudo-Spectrum.** $\theta_1, \ldots, \theta_K$ are the unique zeros of the function

$$\eta(\theta) = \|\boldsymbol{\Pi}\mathbf{a}(\theta)\|_2^2 = \mathbf{a}(\theta)^* \left(\mathbf{I} - \sum_{k=1}^{K} \mathbf{u}_k \mathbf{u}_k^*\right) \mathbf{a}(\theta).$$

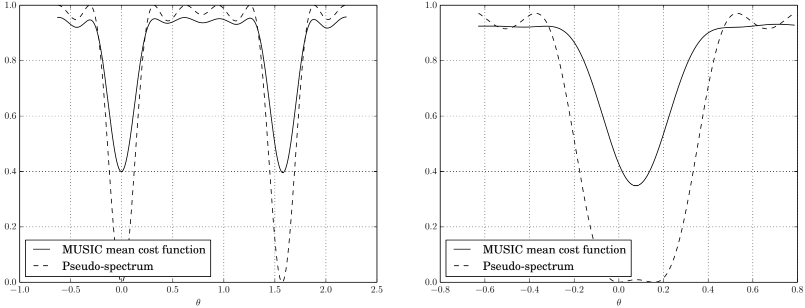
The MUSIC method (2)

The MUSIC method [Schmidt'79]

Estimate $\theta_1, \ldots, \theta_K$ as the K deepest local minimizers $\hat{\theta}_{1,N}, \ldots, \hat{\theta}_{K,N}$ of

$$\hat{\eta}_N(\theta) = \left\| \hat{\mathbf{\Pi}}_N \mathbf{a}(\theta) \right\|_2^2$$
$$= \mathbf{a}(\theta)^* \left(\mathbf{I} - \sum_{k=1}^{K} \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \right) \mathbf{a}(\theta)$$

The MUSIC method (3)



(a) $\theta_1 = 0, \theta_2 = 5 \times \frac{2\pi}{M}$ (b) $\theta_1 = 0, \theta_2 = \frac{\pi}{M}$

Figure : $\mathbb{E}[\hat{\eta}_N(\theta)]$, pseudo-spectrum $\eta(\theta)$ for $M = 20$, $N = 40$, $\sigma = 1$, $\mathbf{\Gamma} = \mathbf{I}$

The MUSIC method (4)

- **Consistency.** In the large sample size regime $N \to \infty$, the LLN implies

$$\left\| \hat{\mathbf{\Pi}}_N - \mathbf{\Pi} \right\|_2 \xrightarrow[N \to \infty]{a.s.} 0,$$

and thus

$$\sup_{\theta \in [-\pi, \pi]} |\hat{\eta}_N(\theta) - \eta(\theta)| \xrightarrow[N \to \infty]{a.s.} 0 \text{ and } \hat{\theta}_{k,N} \xrightarrow[N \to \infty]{a.s.} \theta_k.$$

- **Asymptotic normality.** In [Stoica-Nehorai'89], it was shown that

$$\sqrt{N} \left(\hat{\theta}_{k,N} - \theta_k \right) \xrightarrow[N \to \infty]{\mathcal{D}} \mathcal{N}\left(0, \omega_k^2\right)$$

with

$$\omega_k^2 = \frac{\sigma^2}{2 \left\| \mathbf{\Pi} \mathbf{a}'(\theta_k) \right\|_2^2} \sum_{\ell=1}^{K} \frac{\lambda_\ell \left| \mathbf{a}(\theta_k)^* \mathbf{u}_\ell \right|^2}{(\lambda_\ell - \sigma^2)^2}.$$

The MUSIC method (5)

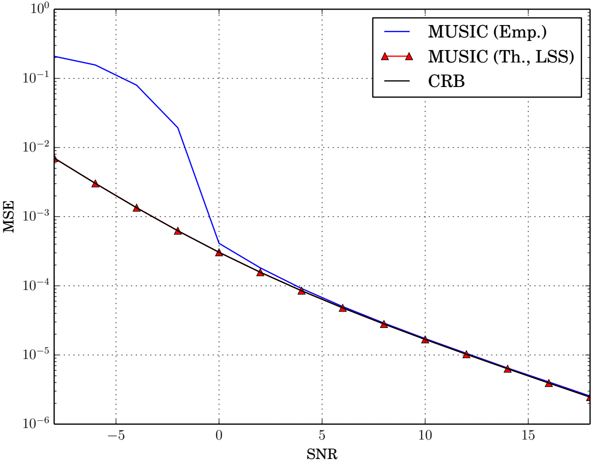


Figure : MSE of $\hat{\theta}_{1,N}$ (MUSIC) and CRB for $M = 20$ and $N = 100$, $\theta_1 = 0$, $\theta_2 = 5 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR = -$10\log(\sigma^2)$.

Large dimensional regime - Spectral projections (1)

- **Context.** $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R}_N)$, with $\mathbf{R}_N$ having eigenvalues $\lambda_{1,N} \geq \ldots \geq \lambda_{K,N} > \sigma^2$ (mult. $M-K$) such that K si fixed w.r.t. N.
- **Detectability condition.** The K sources are detectable if for all $k \in \{1, \ldots, K\}$,

$$\lambda_{k,N} \xrightarrow[M,N\to\infty]{} \lambda_k > \sigma^2\left(1+\sqrt{c}\right).$$

We assume this condition from now on.

- Behaviour of the spectral projections $\hat{\mathbf{u}}_{k,N}\hat{\mathbf{u}}_{k,N}^*$ and $\hat{\mathbf{\Pi}}_N$?

Due to the increasing dimension, we consider sesquilinear forms $\mathbf{d}_{1,N}^*\hat{\mathbf{u}}_{k,N}\hat{\mathbf{u}}_{k,N}^*\mathbf{d}_{2,N}$.

Large dimensional regime - Spectral projections (2)

Theorem [Paul'07]

Let $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R}_N)$, with $\mathbf{R}_N$ having eigenvalues $\lambda_{1,N} \geq \ldots \geq \lambda_{K,N} > \sigma^2$ (mult. $M-K$) such that K is fixed w.r.t. N.

If, for $k \in \{1, \ldots, K\}$, $\lambda_{k,N} \xrightarrow[M,N\to\infty]{} \lambda_k$ and

$$\lambda_1 > \ldots > \lambda_K > \sigma^2 \left(1 + \sqrt{c}\right),$$

then for all deterministic unit norm vectors $\mathbf{d}_{1,N}, \mathbf{d}_{2,N}$,

$$\mathbf{d}_{1,N}^* \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \mathbf{d}_{2,N} - h_{\sigma^2,c}(\lambda_k) \mathbf{d}_{1,N}^* \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \mathbf{d}_{2,N} \xrightarrow[M,N\to\infty]{a.s.} 0,$$

where

$$h_{\sigma^2,c}(\lambda) = \frac{\left(\lambda - \sigma^2\right)^2 - \sigma^4 c}{\left(\lambda - \sigma^2\right)\left(\lambda - \sigma^2(1-c)\right)}.$$

Large dimensional regime - Spectral projections (3)

- **Remark 1.** Natural extension when multiplicity of λ_k greater than 1.
- **Remark 2.** $\mathbf{d}_{1,N}^* \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \mathbf{d}_{2,N}$ is an asymptotically biased estimator of $\mathbf{d}_{1,N}^* \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \mathbf{d}_{2,N}$ due to the factor $h_{\sigma^2,c}(\lambda_k)$. Moreover,

$$h_{\sigma^2,c}(\lambda_k) \approx 1 \quad \text{if} \quad c \approx 0 \quad \text{or} \quad \sigma^2 \approx 0.$$

- **Corollary 1.** Setting $\mathbf{d}_{1,N} = \mathbf{d}_{2,N} = \mathbf{u}_{\ell,N}$, we have for all $k, \ell \in \{1, \ldots, K\}$,

$$\left| \hat{\mathbf{u}}_{k,N}^* \mathbf{u}_{\ell,N} \right|^2 = h_{\sigma^2,c}(\lambda_k) \delta_{k-\ell} + o(1).$$

- **Corollary 2.** Concerning the noise subspace projection,

$$\mathbf{d}_{1,N}^* \hat{\mathbf{\Pi}}_N \mathbf{d}_{2,N} = \mathbf{d}_{1,N}^* \mathbf{\Pi}_N \mathbf{d}_{2,N} + \sum_{k=1}^{K} \left(1 - h_{\sigma^2,c}(\lambda_k)\right) \mathbf{d}_{1,N}^* \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \mathbf{d}_{2,N}.$$

MUSIC in the large dimensional regime (1)

Asymptotic behaviour of the MUSIC cost function [Mestre-Lagunas'08]

Under the conditions of the previous theorem, it holds

$$\sup_{\theta \in [-\pi,\pi]} |\hat{\eta}_N(\theta) - \bar{\eta}_N(\theta)| \xrightarrow[M,N\to\infty]{a.s.} 0$$

where the asymptotic equivalent $\bar{\eta}_N(\theta)$ is given by

$$\bar{\eta}_N(\theta) = \underbrace{\eta_N(\theta)}_{\text{Pseudo-spectrum}} + \underbrace{\sum_{k=1}^{K} \left(1 - h_{\sigma^2,c}(\lambda_k)\right) \left|\mathbf{a}(\theta)^* \mathbf{u}_{k,N}\right|^2}_{\text{Bias}}.$$

What is the impact of this bias on the DoA estimates ?

MUSIC in the large dimensional regime (2)

- **Widely spaced DoA.** If $\theta_1, \ldots, \theta_K$ are fixed w.r.t. M, N, and $\mathcal{I}_k$ is a compact interval of $[-\pi, \pi]$ enclosing only θ_k, we can show that

$$\sup_{\theta \in \mathcal{I}_k} \left| \bar{\eta}_N(\theta) - \left(1 - \chi_{k,N} \left|\mathbf{a}(\theta)^* \mathbf{a}(\theta_k)\right|^2\right) \right| \xrightarrow[M,N \to \infty]{} 0,$$

  with $\chi_{k,N}$ bounded away from 0 and 1 as $M, N \to \infty$.

- Function $\theta \mapsto 1 - \chi_{k,N} \left|\mathbf{a}(\theta)^* \mathbf{a}(\theta_k)\right|^2$ has a unique global minimum at θ_K.

- Thus $\hat{\eta}_N(\theta)$ has its K most deepest local minina converging w.p.1 to $\theta_1, \ldots, \theta_K$.

MUSIC in the large dimensional regime (3)

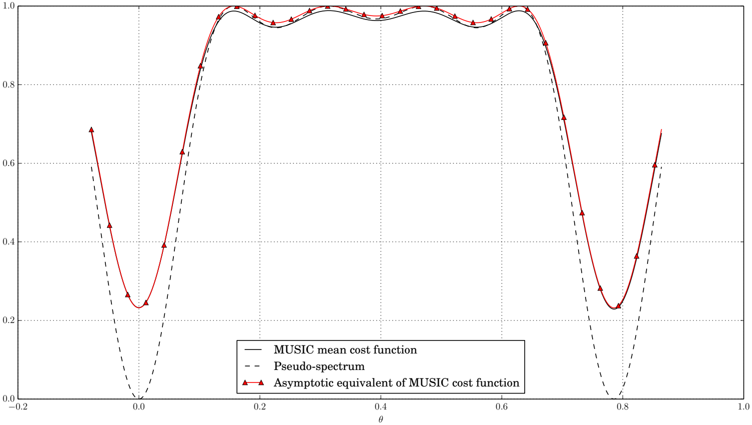


Figure : $\mathbb{E}[\hat{\eta}_N(\theta)]$, pseudo-spectrum $\eta_N(\theta)$ and asymptotic equivalent $\bar{\eta}_N(\theta)$, $M = 40$, $N = 80$, SNR=4 dB, $\mathbf{\Gamma} = \mathbf{I}$, $\theta_1 = 0$, $\theta_2 = 5 \times \frac{2\pi}{M}$.

MUSIC in the large dimensional regime (4)

Performance of MUSIC for widely spaced DoA

Assuming that $K, \theta_1, \ldots, \theta_K$ are fixed with respect to M, N, and that the K sources are detectable. Then we have

$$M\left(\hat{\theta}_{k,N} - \theta_k\right) \xrightarrow[M,N\to\infty]{a.s.} 0.$$

Moreover,

$$N^{3/2}\frac{\hat{\theta}_{k,N} - \theta_k}{\omega_{k,N}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\omega_{k,N}$ depends explicitely on $\lambda_{1,N}, \ldots, \lambda_{K,N}, \sigma^2, \mathbf{u}_{1,N}, \ldots, \mathbf{u}_{M,N}$, and if $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_K)$,

$$\omega_{k,N}^2 \xrightarrow[M,N\to\infty]{} \frac{6\sigma^2(\gamma_k + \sigma^2)}{c^2(\gamma_k^2 - \sigma^4 c)}.$$

MUSIC in the large dimensional regime (5)

- Defining $\rho_k = \frac{\gamma_k}{\sigma^2}$ as the SNR of the k-th source, we have in the uncorrelated case

$$\frac{1}{N^3}\omega_{k,N}^2 \underset{M,N \gg 1}{\approx} \frac{6(1+\rho_k)}{NM^2(\rho_k^2 - c)} \underset{\rho_k \gg 1}{\approx} \frac{6}{NM^2\rho_k}$$

  which coincides with the CRB for large SNR.

- **Spatial periodogram.** We can obtain the same results for the "low resolution" spatial periodogram method which estimates the DoA at the K most significant local maxima of

$$\theta \mapsto \mathbf{a}(\theta)^* \hat{\mathbf{R}}_N \mathbf{a}(\theta).$$

MUSIC in the large dimensional regime (6)

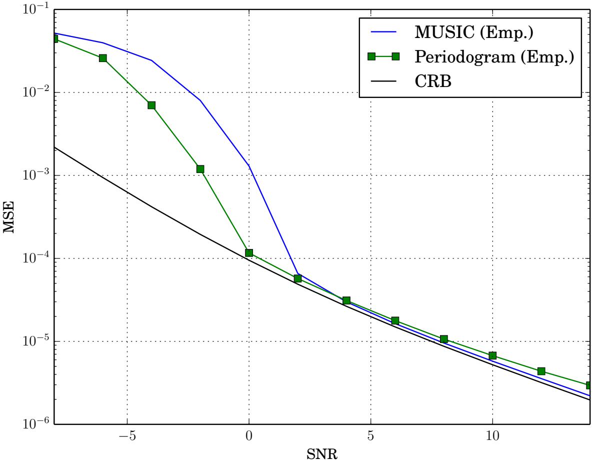


Figure : MUSIC and spatial periodogram for widely spaced DoA, $M = 40, N = 80$, $K = 2$ sources with DoA $\theta_1 = 0$, $\theta_2 = 5 \times \frac{2\pi}{M}$ and $\mathbf{\Gamma} = \mathbf{I}$.

MUSIC in the large dimensional regime (7)

- **Closely spaced DoA.** We assume $K = 2$, $\mathbf{\Gamma} = \mathbf{I}$ and

$$\theta_{2,N} = \theta_{1,N} + \frac{\alpha}{M}, \quad \alpha > 0.$$

- In this case, we have

$$\lambda_{1,N} \xrightarrow[M,N\to\infty]{} \lambda_1 = 1 + |\mathrm{sinc}(\alpha/2)| + \sigma^2$$
$$\lambda_{2,N} \xrightarrow[M,N\to\infty]{} \lambda_2 = 1 - |\mathrm{sinc}(\alpha/2)| + \sigma^2.$$

and the detectability threshold is now $|\mathrm{sinc}(\alpha/2)| < 1 - \sigma^2\sqrt{c}$.

- For any compact $\mathcal{K} \subset \mathbb{R}$,

$$\sup_{\beta\in\mathcal{K}} \left| \overline{\eta}_N \left(\theta_{1,N} + \frac{\beta}{M}\right) - \kappa(\beta) \right| \xrightarrow[M,N\to\infty]{} 0,$$

where κ does not have local maxima at $\beta = 0$ or α in general.

MUSIC in the large dimensional regime (8)

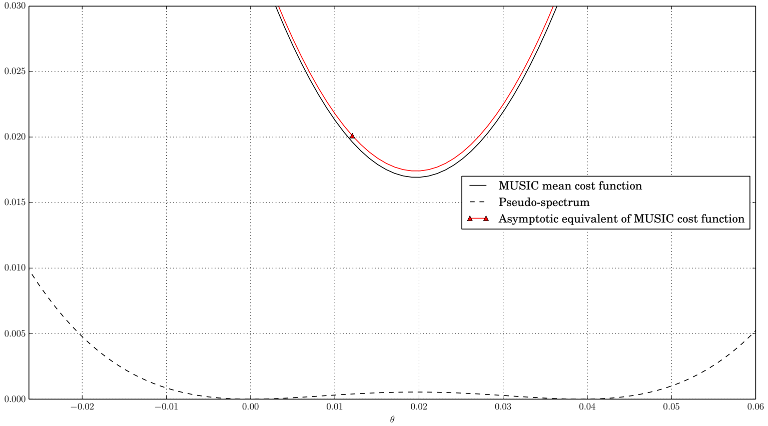


Figure : $\mathbb{E}[\hat{\eta}_N(\theta)]$, pseudo-spectrum $\eta_N(\theta)$ and asymptotic equivalent $\bar{\eta}_N(\theta)$, $M = 40$, $N = 80$, SNR=12 dB, $\mathbf{\Gamma} = \mathbf{I}$ and $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$.

MUSIC in the large dimensional regime (9)

Performance of MUSIC for closely spaced DoA [Vallet et al.'15]

If $K = 2$, $\mathbf{\Gamma} = \mathbf{I}$, and

$$\theta_{2,N} = \theta_{1,N} + \frac{\alpha}{M},$$

where $\alpha > 0$ is such that $|\mathrm{sinc}(\alpha/2)| < 1 - \sigma^2\sqrt{c}$, then for $k \in \{1, 2\}$,

$$\liminf_{M,N\to\infty} M \left|\hat{\theta}_{k,N} - \theta_{k,N}\right| > 0.$$

Failure of MUSIC for closely spaced DoA ...

The G-MUSIC method (1)

- **Reminder.** For $k = 1, \ldots, K$, w.p.1 as $M, N \to \infty$

$$\hat{\lambda}_{k,N} = \phi_{\sigma^2, c_N}(\lambda_{k,N}) + o(1),$$
$$\left|\mathbf{a}(\theta)^* \hat{\mathbf{u}}_{k,N}\right|^2 = h_{\sigma^2, c_N}(\lambda_{k,N}) \left|\mathbf{a}(\theta)^* \mathbf{u}_{k,N}\right|^2 + o(1).$$

where ϕ_{σ^2, c_N}, h_{σ^2, c_N} are defined above, when K is fixed and the detectability condition is satisfied ($\lim \lambda_{K,N} > \sigma^2(1 + \sqrt{c})$).

- **Estimation.**

$$\phi^{-1}_{\sigma^2, c_N}\left(\hat{\lambda}_{k,N}\right) = \lambda_{k,N} + o(1),$$

$$\frac{\left|\mathbf{a}(\theta)^* \hat{\mathbf{u}}_{k,N}\right|^2}{h_{\sigma^2, c_N}\left(\phi^{-1}_{\sigma^2, c_N}\left(\hat{\lambda}_{k,N}\right)\right)} = \left|\mathbf{a}(\theta)^* \mathbf{u}_{k,N}\right|^2 + o(1).$$

The G-MUSIC method (2)

G-MUSIC [Mestre-Lagunas'08]

Define

$$\tilde{\eta}_N(\theta) = 1 - \sum_{k=1}^{K} \frac{\left|\mathbf{a}(\theta)^* \hat{\mathbf{u}}_{k,N}\right|^2}{h_{\sigma^2,c_N}\left(\phi^{-1}_{\sigma^2,c_N}\left(\hat{\lambda}_{k,N}\right)\right)}$$

If K is fixed and the K sources are detectable, it holds that

$$\sup_{\theta \in [-\pi,\pi]} \left|\tilde{\eta}_N(\theta) - \eta_N(\theta)\right| \xrightarrow[M,N\to\infty]{a.s.} 0$$

The G-MUSIC method consists in estimating the DoA as the K deepest local minimizers of $\theta \mapsto \tilde{\eta}_N(\theta)$, denoted in what follows $\tilde{\theta}_{1,N}, \ldots, \tilde{\theta}_{K,N}$.

The G-MUSIC method (3)

- G for generalized (based of Girko's G-estimation ideas)
- **Large sample size.** If $c_N \approx 0$,

$$h_{\sigma^2,c_N}\left(\phi^{-1}_{\sigma^2,c_N}\left(\hat{\lambda}_{k,N}\right)\right) \approx 1,$$

  and

$$\tilde{\eta}_N(\theta) \approx \hat{\eta}_N(\theta).$$

- **High resolution.** Since the asymptotic G-MUSIC cost function is exactly the pseudo-spectrum, the performance is expected to be better for closely spaced DoA.

The G-MUSIC method (4)

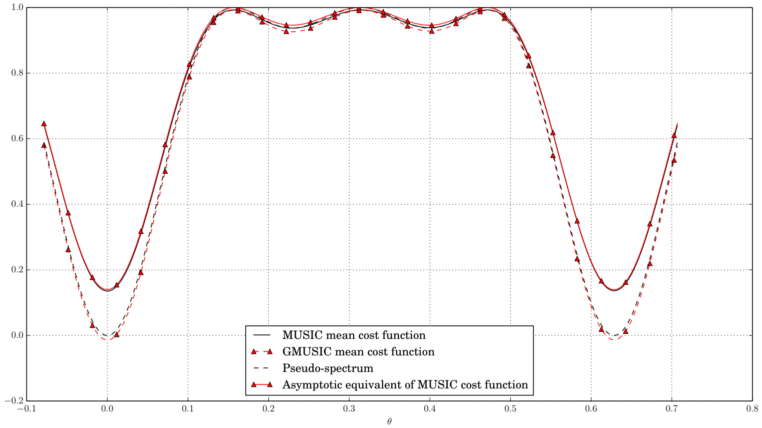


Figure : $\mathbb{E}[\tilde{\eta}_N(\theta)]$,$\mathbb{E}[\hat{\eta}_N(\theta)]$, pseudo-spectrum $\eta_N(\theta)$ and asymptotic equivalent $\bar{\eta}_N(\theta)$, $M = 40$, $N = 80$, SNR=4 dB, $\mathbf{\Gamma} = \mathbf{I}$, $\theta_1 = 0$, $\theta_2 = 4 \times \frac{2\pi}{M}$.

The G-MUSIC method (5)

Performance of G-MUSIC for widely spaced DoA [Vallet et al.'15]

Assuming that $K, \theta_1, \ldots, \theta_K$ are fixed with respect to M, N, and that the K sources are detectable. Then we have

$$M \left(\tilde{\theta}_{k,N} - \theta_k \right) \xrightarrow[M,N\to\infty]{a.s.} 0.$$

Moreover,

$$N^{3/2} \frac{\hat{\theta}_{k,N} - \theta_k}{\omega_{k,N}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\omega_{k,N}$ depends explicitely on $\lambda_{1,N}, \ldots, \lambda_{K,N}, \sigma^2, \mathbf{u}_{1,N}, \ldots, \mathbf{u}_{M,N}$, and if $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_K)$,

$$\omega_{k,N}^2 \xrightarrow[M,N\to\infty]{} \frac{6\sigma^2(\gamma_k + \sigma^2)}{c^2(\gamma_k^2 - \sigma^4 c)}.$$

The G-MUSIC method (6)

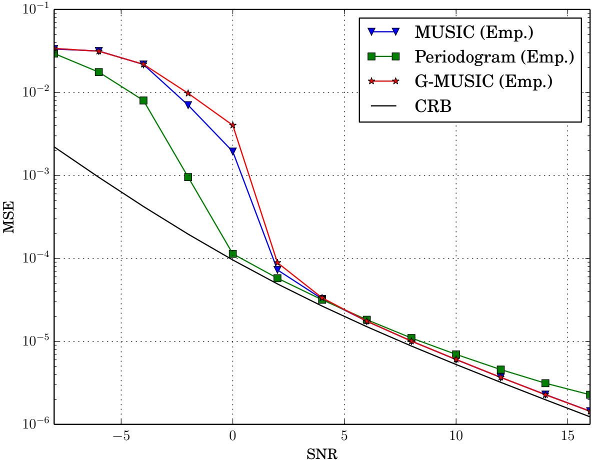


Figure : MSE of DoA estimate of θ_1 for G-MUSIC, MUSIC and spatial periodograms, for $M = 20$ and $N = 100$, $\theta_1 = 0$, $\theta_2 = 5 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR = -10 $\log(\sigma^2)$.

The G-MUSIC method (7)

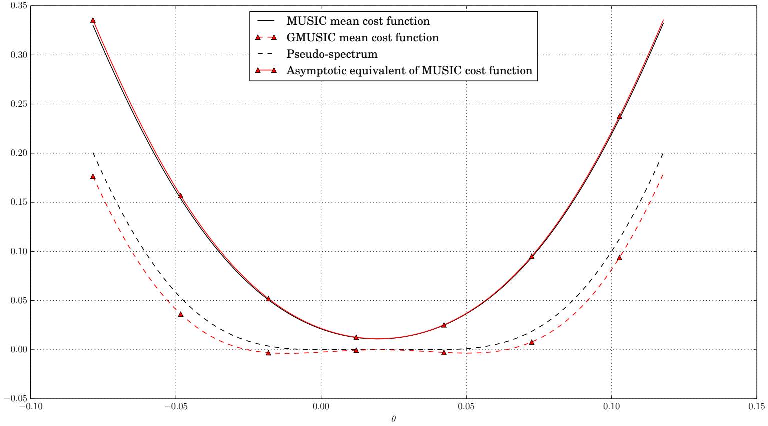


Figure : $\mathbb{E}[\tilde{\eta}_N(\theta)]$,$\mathbb{E}[\hat{\eta}_N(\theta)]$, pseudo-spectrum $\eta_N(\theta)$ and asymptotic equivalent $\bar{\eta}_N(\theta)$, $M = 40$, $N = 80$, SNR=14 dB, $\mathbf{\Gamma} = \mathbf{I}$ and $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$.

The G-MUSIC method (8)

Performance of G-MUSIC for closely spaced DoA [Vallet et al.'15]

If $K = 2$, $\mathbf{\Gamma} = \mathbf{I}$, and

$$\theta_{2,N} = \theta_{1,N} + \frac{\alpha}{M},$$

where $\alpha > 0$ is such that $|\mathrm{sinc}(\alpha/2)| < 1 - \sigma^2\sqrt{c}$, then for $k \in \{1, 2\}$,

$$M \left|\hat{\theta}_{k,N} - \theta_{k,N}\right| \xrightarrow[M,N\to\infty]{a.s.} 0.$$

Moreover,

$$N^{3/2} \frac{\hat{\theta}_{k,N} - \theta_k}{\omega_{k,N}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\omega_{k,N}$ depends explicitely on $\lambda_{1,N}, \ldots, \lambda_{K,N}, \sigma^2, \mathbf{u}_{1,N}, \ldots, \mathbf{u}_{M,N}$, and

The G-MUSIC method (9)

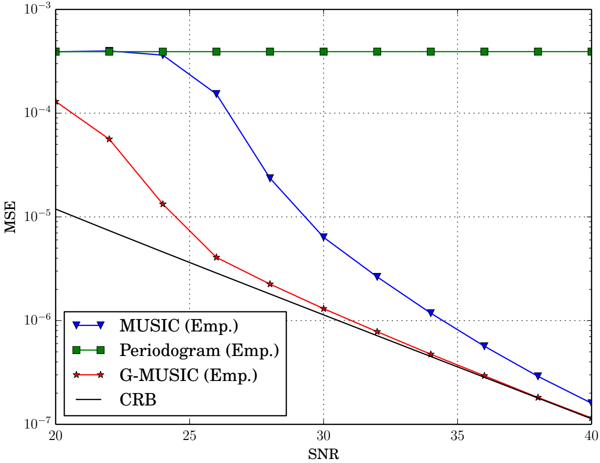


Figure : MSE of DoA estimate of θ_1 for G-MUSIC, MUSIC and spatial periodograms, for $M = 40$ and $N = 80$, $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR $=$ -$10\log(\sigma^2)$.

The G-MUSIC method (10)

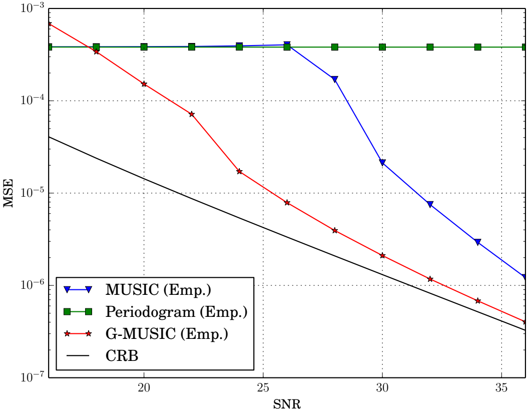


Figure : MSE of DoA estimate of θ_1 for G-MUSIC, MUSIC and spatial periodograms, for $M = 40$ and $N = 80$, $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = [1, 0.5; 0.5, 1]$, against SNR = -$10 \log(\sigma^2)$.

The G-MUSIC method (11)

- **Outlier probability.** $P_{\mathrm{OUT}} = \mathbb{P}\left(\bigcup_{k=1}^{2}\left\{\left|\tilde{\theta}_k - \theta_k\right| > \frac{|\theta_1 - \theta_2|}{2}\right\}\right)$

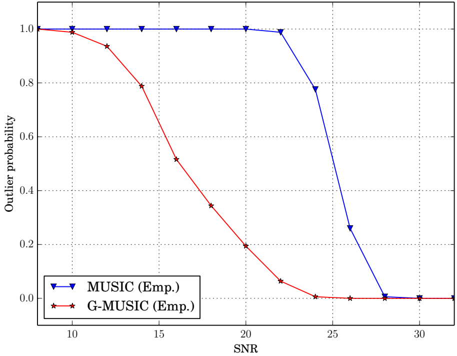


Figure : Outlier probability for GMUSIC and MUSIC, with $M = 40$ and $N = 80$, $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR = $-10\log(\sigma^2)$.

Two ways to get rid off the detectability condition (1)

- **G-MUSIC drawback.** The main limitation of G-MUSIC lies in the K source detectability condition: for all $k \in \{1, \ldots, K\}$,

$$\lambda_{k,N} \xrightarrow[M,N\to\infty]{} \lambda_k > \sigma^2(1+\sqrt{c}),$$

  which requires a sufficiently large SNR.

- **Solution 1.** Decrease c, that is, reduce the dimension M, or in the best case, trade sensors for samples.

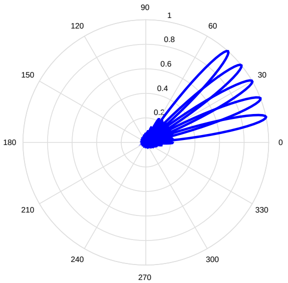
- **Solution 2.** Estimate consistently the covariance $\mathbf{R}_N$ in the large dimensional regime, i.e. find an estimator $\tilde{\mathbf{R}}_N$ of $\mathbf{R}_N$ such that

$$\left\| \tilde{\mathbf{R}}_N - \mathbf{R}_N \right\|_2 \xrightarrow[M,N\to\infty]{a.s.} 0.$$

Two ways to get rid off the detectability condition (2)

- **Beamspace MUSIC.** Prefiltering the data to focus the array onto an angular sector Θ where the DoA are located, before applying MUSIC.
- **DFT Beamformer.** Form L orthonormal beams $\mathbf{a}(\psi_{1,N}), \ldots, \mathbf{a}(\psi_{L,N})$ with

$$\{\psi_1, \ldots, \psi_L\} = \left\{-\pi + \frac{2\pi(m-1)}{M} : m = 1, \ldots, M\right\} \cap \Theta.$$



Two ways to get rid off the detectability condition (3)

- **Filtered model.** New samples $\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_N$ i.i.d. with

$$\tilde{\mathbf{y}}_n = \mathbf{B}_N^* \mathbf{y}_n$$

$$= \tilde{\mathbf{A}} \mathbf{s}_n + \tilde{\mathbf{v}}_n,$$

  where

  - $\mathbf{B} = [\mathbf{a}(\psi_1), \ldots, \mathbf{a}(\psi_L)]$ (beamforming matrix)
  - $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \ldots, \tilde{\mathbf{a}}(\theta_K)]$, with $\tilde{\mathbf{a}}(\theta) = \mathbf{B}^* \mathbf{a}(\theta)$.
  - $\tilde{\mathbf{v}}_n = \mathbf{B}^* \mathbf{v}_n \sim \mathcal{N}_{\mathbb{C}^L}(\mathbf{0}, \sigma^2 \mathbf{I})$

- **New SCM.** $\tilde{\mathbf{R}}_N = \frac{1}{N} \sum_{n=1}^N \tilde{\mathbf{y}}_n \tilde{\mathbf{y}}_n^*$.

Two ways to get rid off the detectability condition (4)

Beamspace MUSIC algorithm [Forster-Vezzosi'87]

Estimate the DoA as the K deepest minima of

$$\theta \mapsto \left\| \tilde{\mathbf{\Pi}}_N \tilde{\mathbf{a}}(\theta) \right\|_2^2,$$

where $\tilde{\mathbf{\Pi}}_N$ is the noise projector estimate based on the new samples $\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_N$.

Two ways to get rid off the detectability condition (5)

- **Dimensionality reduction 1.** If Θ is fixed w.r.t. M, N (L scales with M, N)

$$\frac{L}{N} \xrightarrow[M,N\to\infty]{} d = \frac{|\Theta|}{2\pi} c \leq c.$$

The minimal SNR for source detectability decreases.

- **Dimensionality reduction 2.** If L is fixed w.r.t. M, N (thus $|\Theta| = \mathcal{O}\left(\frac{1}{M}\right)$)

The detectability condition disappears and we can recover consistency with rate $o\left(\frac{1}{M}\right)$ in a closely spaced DoA scenario.

Two ways to get rid off the detectability condition (6)

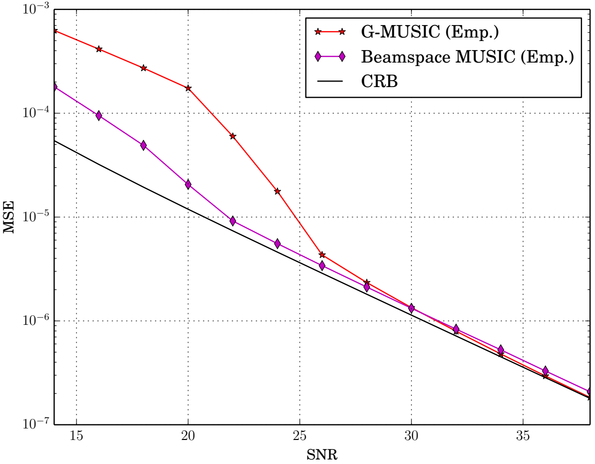


Figure : MSE of DoA estimate of θ_1 for Beamspace-MUSIC, G-MUSIC, for $M = 20$ and $N = 100$, $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR = -$10\log(\sigma^2)$ and focusing sector s.t. $|\Theta| = 10 * |\theta_2 - \theta_1|$.

Two ways to get rid off the detectability condition (7)

- **SCM drawback.** In the case of ULA, the covariance matrix

$$\mathbf{R} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}(\theta_k)^* + \sigma^2 \mathbf{I}$$

  is Toeplitz while the SCM

$$\hat{\mathbf{R}}_N = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^*$$

  is not.

Two ways to get rid off the detectability condition (8)

- **Toeplitz rectification.** To improve the estimation of $\mathbf{R}$, one can use the orthogonal projection of $\hat{\mathbf{R}}_N$ onto the space $\mathcal{T}$ of Toeplitz matrices:

$$\tilde{\mathbf{R}}_N = \pi_{\mathcal{T}}\left(\hat{\mathbf{R}}_N\right),$$

where

$$\pi_{\mathcal{T}}(\mathbf{X}) = \sum_{m=-(M-1)}^{M-1} \operatorname{tr}\left(\mathbf{X}\mathbf{E}_m^*\right)\mathbf{E}_m, \quad \mathbf{E}_m = \frac{1}{\sqrt{M-|m|}}\mathbf{J}^m$$

and

$$\mathbf{J} = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix}, \quad \mathbf{J}^{-1} := \mathbf{J}^* = \begin{pmatrix} & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

Two ways to get rid off the detectability condition (9)

R-MUSIC [Cazdow'87, Forster'01]

Estimate the DoA as the K deepest minimizers $\tilde{\theta}_{1,N}, \ldots, \tilde{\theta}_{K,N}$ of

$$\theta \mapsto \tilde{\eta}_N(\theta) = \left\| \tilde{\mathbf{\Pi}}_N \mathbf{a}(\theta) \right\|_2^2,$$

where $\tilde{\mathbf{\Pi}}_N$ is the noise projector estimate based on the rectified SCM $\tilde{\mathbf{R}}_N$.

Two ways to get rid off the detectability condition (10)

Performance of R-MUSIC [Vallet-Loubaton'17]

The following assertions hold:

- $\left\| \tilde{\mathbf{R}}_N - \mathbf{R}_N \right\|_2 \xrightarrow[M,N\to\infty]{a.s.} 0.$
- $\sup_\theta |\tilde{\eta}_N(\theta) - \eta_N(\theta)| \xrightarrow[M,N\to\infty]{a.s.} 0.$
- For widely/closely spaced DoA scenarios introduced above,

$$M \left| \tilde{\theta}_{k,N} - \theta_k \right| \xrightarrow[M,N\to\infty]{a.s.} 0.$$

Two ways to get rid off the detectability condition (11)

- **Remark.** The operator norm consistency of $\tilde{\mathbf{R}}_N$ holds whatever the order of magnitude of the eigenvalues of $\mathbf{R}_N$ compared to $\sigma^2(1+\sqrt{c})$.
- **CLT.**

$$M^{3/2}\frac{\tilde{\theta}_{k,N}-\theta_k}{\rho_{k,N}} \xrightarrow[M,N\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

where

$$\rho_{k,N}^2 = \frac{c_N \left\| \mathbf{R}_N^{1/2}\mathbf{T}_{k,N}\mathbf{R}_N^{1/2} \right\|_F^2}{\left\| \mathbf{\Pi}_N \frac{\mathbf{a}'(\theta_k)}{M} \right\|_2^4}.$$

with $\mathbf{T}_{k,N}$ independent of σ^2 (explicitly known).

$\Rightarrow$ MSE stagnation for large SNR

Two ways to get rid off the detectability condition (12)

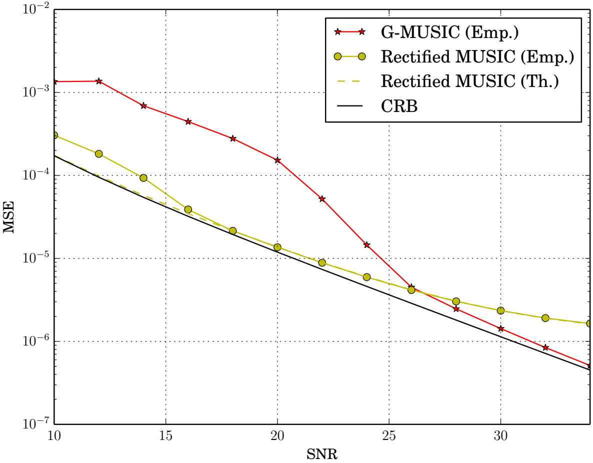


Figure : MSE of DoA estimate of θ_1 for Rectified-MUSIC, G-MUSIC, for $M = 40$ and $N = 80$, $\theta_1 = 0$, $\theta_2 = 0.25 \times \frac{2\pi}{M}$, $\mathbf{\Gamma} = \mathbf{I}$, against SNR = $-10\log(\sigma^2)$

Contents

Large dimensional regime - non-fixed rank (1)

- In general, $\mathbf{R}$ is not a small rank perturbation of $\sigma^2\mathbf{I}$.
- **Motivation 1.** The number of sources K may not be small compared to M.
- **Motivation 2.** In the context of clutter/jammers,

$$\mathbf{R} = \mathbf{A}\mathbf{\Gamma}\mathbf{A}^* + \mathbf{C} + \sigma^2\mathbf{I},$$

where $\mathbf{C} = M\int_{-\pi}^{\pi} \mathbf{a}(\theta)\mathbf{a}(\theta)^* \mathrm{d}\nu(\theta)$ with ν a certain measure representing the spatial energy distribution of the clutter. For example, if $\mathrm{d}\nu(\theta) = f(\theta)\mathrm{d}\theta$, with $\mathrm{supp}(f) = [\theta_-, \theta_+] \subset (-\pi, \pi)$ and f continuous on (θ_-, θ_+), then

$$\frac{\mathrm{rank}(\mathbf{C})}{M} \xrightarrow[M\to\infty]{} 1 - \frac{\theta_+ - \theta_-}{2\pi}.$$

Large dimensional regime - non-fixed rank (2)

Theorem [Silverstein-Bai'95]

If $\mathbf{y}_1, \ldots, \mathbf{y}_N$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}(\mathbf{0}, \mathbf{R}_N)$, with $\limsup \|\mathbf{R}_N\| < \infty$ as $M, N \to \infty$. Then with probability one,

$$\hat{\mu}_N - \mu_N \xrightarrow[M,N\to\infty]{w} 0$$

where μ_M is a deterministic probability measure given through its Stieltjes transform

$$m_{\mu_N}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_N(\lambda)}{\lambda - z},$$

which satisfies the following equation for all $z \in \mathbb{C}\backslash\mathbb{R}$:

$$m_{\mu_N}(z) = \frac{1}{M}\mathrm{tr}\left(\mathbf{R}_N(1 - c_N - c_N z m_{\mu_N}(z)) - z\mathbf{I}\right)^{-1}.$$

Large dimensional regime - non-fixed rank (3)

- **Density.** μ_N admits a density with compact support given by [Silverstein-Choi'95]

$$\frac{\mathrm{d}\mu_N(\lambda)}{\mathrm{d}\lambda} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im}\left(m_{\mu_N}(\lambda + \mathrm{i}\epsilon)\right).$$

- **Marcenko-Pastur distribution.** When $\mathbf{R}_N = \sigma^2 \mathbf{I}$, $m_{\mu_N}(z)$ is solution to the quadratic equation

$$m_{\mu_N}(z) = \frac{1}{\sigma^2 \left(1 - c_N - c_N z m_{\mu_N}(z)\right) - z}$$

and admits an analytical expression, from which the Marcenko-Pastur distribution is obtained.

Large dimensional regime - non-fixed rank (4)

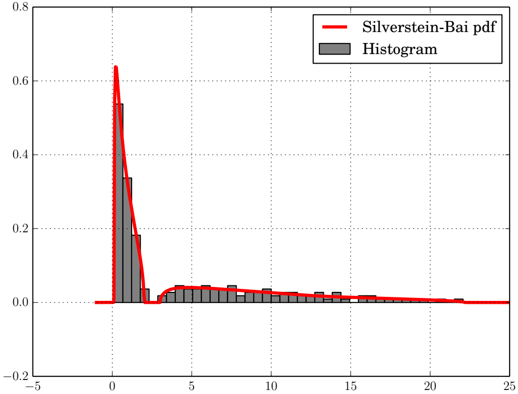


Figure : Silverstein-Bai distribution and histogram of the sample eigenvalues for $M = 200$, $N = 400$, and $\mathbf{R}$ having eigenvalues $1, 8, 13$ with proportions $\frac{6}{10}, \frac{3}{10}, \frac{1}{10}$.

Large dimensional regime - non-fixed rank (5)

- **Support separation.** In general, the support of μ_N splits in several "clusters" with each eigenvalue of $\mathbf{R}_N$ being related to a cluster.

- **Detectability condition.** If $\sigma^2 = \lambda_{K+1,N} = \ldots = \lambda_{M,N}$ is sufficiently spaced from $\lambda_{1,N}, \ldots, \lambda_{K,N}$, then the first cluster is related to the eigenvalue σ^2 and splits from the others [Mestre'08].

- **Separation of the sample eig.** In that case, with probability one,

$$\hat{\lambda}_{K+1,N}, \ldots, \hat{\lambda}_{M,N} \in \left(\lambda^-, \lambda^+\right)$$

  for all large M, N, while $\liminf \hat{\lambda}_{K+1,N} > \lambda^+$, where (λ^-, λ^+) is any fixed open interval enclosing only the first cluster.

Large dimensional regime - non-fixed rank (6)

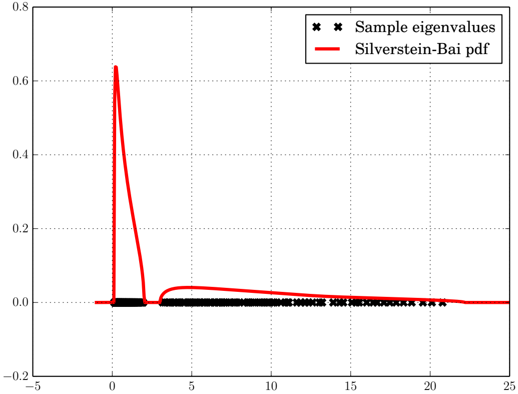


Figure : Location of the sample eigenvalues w.r.t. the Silverstein-Bai distribution, for $M = 200$, $N = 400$, and $\mathbf{R}$ having eigenvalues $1, 8, 13$ with proportions $\frac{6}{10}, \frac{3}{10}, \frac{1}{10}$

Large dimensional regime - non-fixed rank (7)

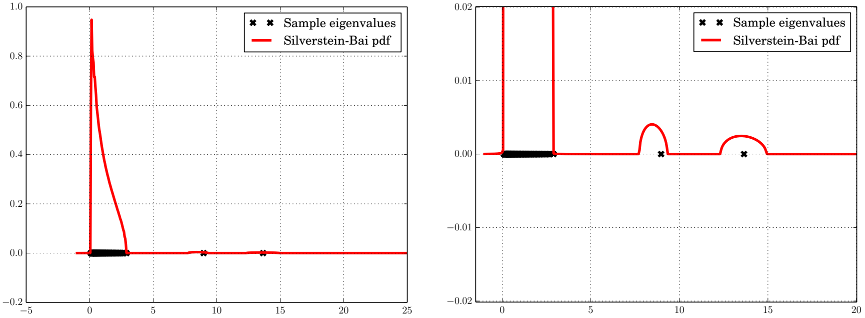


Figure : Location of the sample eigenvalues w.r.t. the Silverstein-Bai distribution, for $M = 200$, $N = 400$, and $\mathbf{R}$ having eigenvalues $1, 8, 13$ with proportions $\frac{198}{200}, \frac{1}{200}, \frac{1}{200}$

Large dimensional regime - non-fixed rank (8)

- [Mestre-Lagunas'08] G-MUSIC in its most generality, assuming the source number K dependent of M, N.
- [Vinogradova et al.'13] Detection in spatially correlated noise, DoA estimation in temporally correlated noise.
- [Najim et al.'16] Performance of MUSIC in the presence of spatially spread clutter.
- [Mestre-Vallet'17] Signal detection through coherence tests.
- [Combernoux et al.'15] Performance of LR-ANMF detector.

Towards wideband array processing in large dimensions (1)

- **Wideband model.** When considering uncorrelated wideband source signals, the covariance matrix of the obserbations writes

$$\mathbf{R} = \underbrace{\sum_{k=1}^{K} \int_{-1/2}^{1/2} \mathbf{b}_M(\theta_k, \nu + \nu_c)\mathbf{b}_M(\theta_k, \nu + \nu_c)^* \mathrm{d}\varrho_k(\nu)}_{\mathbf{R}_s} + \sigma^2 \mathbf{I},$$

where ϱ_k is the spectral measure of the k-th source and

$$\mathbf{b}_M(\theta, \nu) = \Big(1, \exp\left(\mathrm{i}C\theta\nu\right), \ldots, \exp\left(\mathrm{i}C(M-1)\theta\nu\right)\Big)^T,$$

with $C > 0$ a constant and ν_c the carrier frequency renormalized by the sampling frequency.

In general, $\mathbf{R}_s$ is not rank-deficient nor has well separated signal/noise subspaces.

Towards wideband array processing in large dimensions (2)

- **Spatio-temporal covariance matrix.** To increase the dimensionality, a standard technique [Bienvenu'83] consists in building the $M \times L$ stacked vectors

$$\mathbf{y}_n^{(L)} = (y_{1,n}, \ldots, y_{1,n+L-1}, \ldots, y_{M,n}, \ldots, y_{M,n+L-1})^T$$

and $\mathbf{R}^{(L)} = \mathbb{E}\left[\mathbf{y}_n^{(L)} \mathbf{y}_n^{(L)*}\right] = \mathbf{R}_s^{(L)} + \sigma^2 \mathbf{I}_{ML}$,

$$\mathbf{R}_s^{(L)} = \sum_{k=1}^{K} \int_{-1/2}^{1/2} (\mathbf{b}_M(\theta, \nu + \nu_c) \otimes \mathbf{b}_L(\nu)) (\mathbf{b}_M(\theta, \nu + \nu_c) \otimes \mathbf{b}_L(\nu))^* \, \mathrm{d}\varrho_k(\nu)$$

with $\mathbf{b}_L(\nu) = \Big(1, \exp(\mathrm{i}\nu), \ldots, \exp(\mathrm{i}(M-1)\nu)\Big)^T$.

- As $L \to \infty$ while M is fixed, a proportion of the eigenvalues of $\mathbf{R}_s^{(L)}$ related to the K sources bandwidth split from the other ones which converge to 0.

Towards wideband array processing in large dimensions (3)

- **Estimation.** The $ML \times ML$ spatio-temporal covariance matrix $\mathbf{R}^{(L)}$ is usually estimated empirically by

$$\hat{\mathbf{R}}_N^{(L)} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n^{(L)} \mathbf{y}_n^{(L)*}.$$

- **Large dimensional regime.** Behaviour of the eigenvalues/eigenvectors of $\hat{\mathbf{R}}_N^{(L)}$ in the regime where $M, L, N \to \infty$?

- $\mathbf{y}_1^{(L)}, \ldots, \mathbf{y}_{N+L-1}^{(L)}$ are not i.i.d. (matrix $\mathbf{Y}_N^{(L)} = \left[\mathbf{y}_1^{(L)}, \ldots, \mathbf{y}_{N+L-1}^{(L)}\right]$ has a block-Hankel structure), and new results for this model are needed.

Towards wideband array processing in large dimensions (4)

Theorem [Loubaton'16]

Let $\mathbf{y}_1, \ldots, \mathbf{y}_{N+L-1}$ i.i.d. $\mathcal{N}_{\mathbb{C}^M}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right)$ and $\hat{\mu}_N$ the e.s.d. of matrix $\hat{\mathbf{R}}_N^{(L)}$.
Assume $M = M(N), L = L(N)$ s.t. $d_N := \frac{ML}{N} \to d > 0$ as $N \to \infty$.

- With probability one,

$$\hat{\mu}_N \xrightarrow[N\to\infty]{w} \mu_{\sigma^2,d},$$

  where $\mu_{\sigma^2,d}$ is the Marcenko-Pastur distribution with scale parameter d.
- If moreover $L = \mathcal{O}\left(N^\alpha\right)$ with $\alpha < \frac{2}{3}$, then

$$\hat{\lambda}_{1,N}^{(L)} \xrightarrow[N\to\infty]{a.s.} \sigma^2\left(1+\sqrt{d}\right)^2 \text{ and } \hat{\lambda}_{ML,N}^{(L)} \xrightarrow[N\to\infty]{a.s.} \sigma^2\left(1-\sqrt{d}\right)^2$$

  where $\hat{\lambda}_{1,N}^{(L)} \geq \ldots \geq \hat{\lambda}_{M,N}^{(L)}$ are the eigenvalues of $\hat{\mathbf{R}}_N^{(L)}$.

Towards wideband array processing in large dimensions (5)

- [Pham-Loubaton.'15] Test detection in the context of multipath channels (sum of largest eigenvalues over the trace)
- [Pham et al.'16] Analysis of the spatial smoothing on the MUSIC method (narrowband model, but involves block-Hankel observations matrices)
- [Pham-Loubaton'16] Optimization of the loading factor of trained spatio-temporal Wiener filters

Other works

- **Robust array processing.** [Couillet et al.'15], [Couillet'15]
- **Capacity of MIMO systems.** [Telatar'99], [Chuah et al.'02], [Tulino et al.'05], [Hachem et al.'08], ...

Open problems in large dimensional array processing

- Wideband array processing
- Analysis of ESPRIT like methods
- Higher-order detection and subspace methods, blind source separation methods
- Parametric detection

Contents

Conclusion

- Standard analysis of array processing methods based on large sample size $N \gg 1$ is not reliable in practice when the number of sensors M is s.t.

$$M \approx N.$$

- The double asymptotic regime

$$M, N \to \infty, \frac{M}{N} \to c > 0$$

is better suited to model this situation.

- Large random matrix results provide accurate results on the behaviour of eigenvalues/eigenvectors of the SCM to analyze standard detection/DoA estimation methods, and to develop improved algorithms.

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