Méthodologies d'Estimation et de Détection Robuste en Conditions Non-Standards Pour le Traitement d'Antenne, l'Imagerie et le Radar

Jean-Philippe Ovarlez[1,2]

[1]SONDRA, CentraleSupélec, France
[2]French Aerospace Lab, ONERA DEMR/TSI, France

Joint works with F. Pascal, P. Forster, G. Ginolhac, M. Mahot, J. Frontera-Pons, A. Breloy, G. Vasile, and many others

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Part B

Robust Detection and Estimation Schemes

Part B: Contents

Outline

Modeling the background

Let $\mathbf{z}$ be a complex circular random vector of length m. $\mathbf{z}$ has a Complex Elliptically Symmetric (CES) distribution ($CE(\boldsymbol{\mu}, \Sigma, g_.)$) if its PDF is [Kelker, 1970, Frahm, 2004, Ollila et al., 2012]:

$$g_{\mathbf{z}}(\mathbf{z}) = \pi^{-m} |\Sigma|^{-1} h_z((\mathbf{z}-\boldsymbol{\mu})^H \Sigma^{-1} (\mathbf{z}-\boldsymbol{\mu})), \quad (1)$$

where $h_z : [0, \infty) \rightarrow [0, \infty)$ is the density generator, where $\boldsymbol{\mu}$ is the statistical mean (generally known or $= \mathbf{0}$) and Σ is the scatter matrix. In general, $E\left[\mathbf{z}\,\mathbf{z}^H\right] = \alpha\,\Sigma$ where α is known.

- **Large class of distributions**: Gaussian ($h_z(z) = \exp(-z)$, SIRV, MGGD ($h_z(z) = \exp(-z^\alpha)$), etc.
- **Closed under affine transformations** (e.g. matched filter),
- **Stochastic representation theorem**: $\boxed{\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{u}^{(k)}}$,
  where $\mathcal{R} \geq 0$, independent of $\mathbf{u}^{(k)}$ and $\Sigma = \mathbf{A}\mathbf{A}^H$ is a factorization of Σ, where $\mathbf{A} \in \mathbb{C}^{m \times k}$ with $k = \text{rank}(\Sigma)$.

SIRV: a CES subclass

The m-vector $\mathbf{z}$ is a complex Spherically Invariant Random Vector [Yao, 1973, Jay, 2002] if its PDF can be put in the following form:

$$g_{\mathbf{z}}(\mathbf{z}) = \frac{1}{\pi^m |\Sigma|} \int_0^\infty \frac{1}{\tau^m} \exp\left(\frac{(\mathbf{z}-\boldsymbol{\mu})^H \Sigma^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\tau} \right) p_\tau(\tau)\, d\tau\,, \tag{2}$$

where $p_\tau : [0,\infty) \to [0,\infty)$ is the texture generator.

- **Large class of distributions**: Gaussian ($p_\tau(\tau) = \delta(\tau - 1)$), K-distribution ($p_\tau$ gamma), Weibull (no closed form), Student-t (p_τ inverse gamma), etc. Main Gaussian Kernel: closed under affine transformations,
- The texture random scalar is modeling the variation of the power of the Gaussian vector $\mathbf{x}$ along his support (e.g. heterogeneity of the noise along range bins, time, spatial domain, etc.),
- **Stochastic representation theorem**: $\boxed{\mathbf{z} =_d \boldsymbol{\mu} + \sqrt{\tau}\,\mathbf{A}\,\mathbf{x}}$, where $\tau \geq 0$ is the texture, independent of $\mathbf{x}$ and $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \Sigma)$.

Outline

Estimating the covariance matrix: Conventional estimators

Assuming n available SIRV secondary data $\mathbf{z}_k = \sqrt{\tau_k}\, \mathbf{x_k}$ where $\mathbf{x}_k \sim \mathcal{CN}(\mathbf{0}, \Sigma)$ and where τ_k scalar random variable.

- The **Sample Covariance Matrix** (SCM) may be a poor estimate of the Elliptical/SIRV Scatter/Covariance Matrix because of the texture contamination:

$$\hat{\mathbf{S}}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{z}_k\, \mathbf{z}_k^H = \frac{1}{n} \sum_{k=1}^{n} \tau_k\, \mathbf{x}_k\, \mathbf{x}_k^H \neq \frac{1}{n} \sum_{k=1}^{n} \mathbf{x}_k\, \mathbf{x}_k^H \,,$$

- The **Normalized Sample Covariance Matrix** (NSCM) may be a good candidate of the Elliptical SIRV Scatter/Covariance Matrix:

$$\hat{\Sigma}_{NSCM} = \frac{1}{n} \sum_{k=1}^{n} \frac{\mathbf{z}_k\, \mathbf{z}_k^H}{\mathbf{z}_k^H\, \mathbf{z}_k} = \frac{1}{n} \sum_{k=1}^{n} \frac{\mathbf{x}_k\, \mathbf{x}_k^H}{\mathbf{x}_k^H\, \mathbf{x}_k},$$

This estimate does not depend on the texture τ_k but it is biased and share the same eigenvectors but have different eigenvalues, with the same ordering [Bausson et al., 2007].

Estimating the covariance matrix

Let $(\mathbf{z}_1, ..., \mathbf{z}_n)$ be a n-sample $\sim CE_m(\mathbf{0}, \Sigma, g_{\mathbf{z}(.)})$ (Secondary data).

PDF $g_{\mathbf{z}}(.)$ specified: *ML*-estimator of Σ

$$\widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} \frac{-g_{\mathbf{z}}'\left(\mathbf{z}_i^H \, \widehat{\Sigma}^{-1} \, \mathbf{z}_i\right)}{g_{\mathbf{z}}\left(\mathbf{z}_i^H \, \widehat{\Sigma}^{-1} \, \mathbf{z}_i\right)} \, \mathbf{z}_i \, \mathbf{z}_i^H \, ,$$

PDF $g_{\mathbf{z}}(.)$ not specified: *M*-estimator of Σ

$$\widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} u\left(\mathbf{z}_i^H \, \widehat{\Sigma}^{-1} \, \mathbf{z}_i\right) \, \mathbf{z}_i \, \mathbf{z}_i^H \, ,$$

[Maronna et al., 2006, Kent and Tyler, 1991, Pascal, 2006, Pascal et al., 2008a, Pascal et al., 2008b]

- Existence, Uniqueness,
- Convergence of the recursive algorithm, etc.

Examples of M-estimators

SCM:

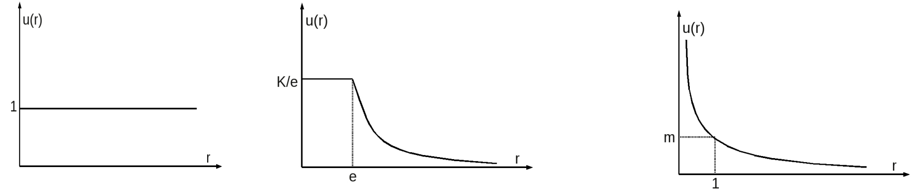
$$u(r) = 1$$

Huber's M-estimator:

$$u(r) = \begin{cases} K/e \text{ if } r <= e \\ K/r \text{ if } r > e \end{cases}$$

FPE (Tyler):

$$u(r) = \frac{m}{r}$$



- Huber = mix between SCM and FPE [Huber, 1964],
- FPE and SCM are "not" (theoretically) M-estimators,
- FPE is the most robust while SCM is the most efficient.

Estimating the covariance matrix: Tyler's M-estimators

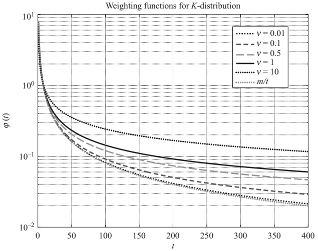
Let $(\mathbf{z}_1, ..., \mathbf{z}_n)$ be a n-sample $\sim CE_m(\mathbf{0}, \Sigma, g_{\mathbf{z}})$ (Secondary data).

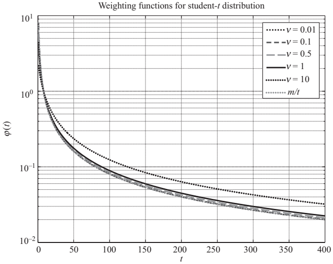
FP Estimate ([Tyler, 1987, Pascal et al., 2008a])

$$\widehat{\Sigma}_{FPE} = \frac{m}{n} \sum_{k=1}^{n} \frac{\mathbf{z}_k \, \mathbf{z}_k^H}{\mathbf{z}_k^H \, \widehat{\Sigma}_{FPE}^{-1} \, \mathbf{z}_k}.$$

- The FPE does not depend on the texture (SIRV or CES distributions),
- Existence, Uniqueness,
- Convergence of the recursive algorithm (identifiability condition: $\mathrm{tr}(\widehat{\Sigma}_{FPE}) = m$),
- True MLE under SIRV distributed noise with unknown deterministic texture $\{\tau_k\}_{k \in [1,n]}$.

Some Weighting Functions of M-estimators





$$\mathbf{u}(t) = \frac{\sqrt{\nu}}{t}\,\frac{K_{\nu-m-1}\left(4\,\nu\,t\right)}{K_{\nu-m}\left(4\,\nu\,t\right)}, \qquad \mathbf{u}(t) = \frac{\nu+2\,m}{\nu+2\,t}\,.$$

We have $\lim\limits_{\nu\to 0}\hat{\Sigma} = \hat{\Sigma}_{FPE}$ and $\lim\limits_{\nu\to\infty}\hat{\Sigma} = \hat{\Sigma}_{SCM}$.

Asymptotic distribution of complex M-estimators

Using the results of Tyler, we derived the following results [Mahot, 2012, Mahot et al., 2013]:

Theorem 1 (Asymptotic distribution of $\hat{\Sigma}$)

$$\sqrt{n}\,\text{vec}(\hat{\Sigma} - \Sigma) \xrightarrow{d} \mathcal{CN}_{m^2}\left(\mathbf{0}_{m^2}, \mathbf{C}, \mathbf{P}\right), \tag{3}$$

where $\mathcal{CN}$ is the complex Gaussian distribution, $\mathbf{C}$ the CM and $\mathbf{P}$ the pseudo CM:

$$\begin{aligned}
\mathbf{C} &= \sigma_1\left(\Sigma^* \otimes \Sigma\right) + \sigma_2\,\text{vec}(\Sigma)\text{vec}(\Sigma)^H,\\
\mathbf{P} &= \sigma_1\left(\Sigma^* \otimes \Sigma\right)\mathbf{K}_{m^2,m^2} + \sigma_2\,\text{vec}(\Sigma)\text{vec}(\Sigma)^T,
\end{aligned}$$

where $\mathbf{K}_{m,m}$ is the $m \times m$ commutation matrix transforming any m-vector $\text{vec}(\mathbf{A})$ into $\text{vec}(\mathbf{A}^T)$ and where the constant σ_1 and σ_1 are completely defined.

An important property of complex M-estimators

- Let $\widehat{\Sigma}$ an estimate of Hermitian positive-definite matrix Σ that satisfies

$$\sqrt{n}\left(\text{vec}(\widehat{\Sigma}-\Sigma)\right) \xrightarrow{d} \mathcal{CN}\left(\mathbf{0}_m, \mathbf{C}, \mathbf{P}\right), \tag{4}$$

with

$$\begin{cases} \mathbf{C} = \nu_1\, \Sigma^* \otimes \Sigma + \nu_2\, \text{vec}(\Sigma)\,\text{vec}(\Sigma)^H, \\ \mathbf{P} = \nu_1\, (\Sigma^* \otimes \Sigma)\, \mathbf{K}_{m^2,m^2} + \nu_2\, \text{vec}(\Sigma)\,\text{vec}(\Sigma)^T, \end{cases}$$

where ν_1 and ν_2 are any real numbers.

e.g.

	SCM	M-estimators	FP
ν_1	1	σ_1	$(m+1)/m$
ν_2	0	σ_2	$-(m+1)/m^2$
...	More accurate		More robust

- Let $H(.)$ be a r-multivariate function on the set of Hermitian positive-definite matrices, with continuous first partial derivatives and such as $H(\mathbf{V}) = H(\alpha\mathbf{V})$ for all $\alpha > 0$, e.g. the ANMF statistic, the MUSIC statistic, etc.

Theorem 2 (Asymptotic distribution of $H(\widehat{\Sigma})$)

$$\sqrt{n}\left(H(\widehat{\Sigma}) - H(\Sigma)\right) \xrightarrow{d} \mathcal{CN}\left(\mathbf{0}_r, \mathbf{C}_H, \mathbf{P}_H\right), \tag{5}$$

where $\mathbf{C}_H$ and $\mathbf{P}_H$ are defined as

$$\begin{aligned}\mathbf{C}_H &= \nu_1\, H'(\Sigma)\,(\Sigma^T \otimes \Sigma)\, H'(\Sigma)^H, \\ \mathbf{P}_H &= \nu_1\, H'(\Sigma)\,(\Sigma^T \otimes \Sigma)\,\mathbf{K}_{m^2,m^2}\, H'(\Sigma)^T,\end{aligned}$$

where $H'(\Sigma) = \left(\dfrac{\partial H(\Sigma)}{\partial \text{vec}(\Sigma)}\right)$.

CES distribution $\Rightarrow$ two-step GLRT ANMF

ANMF test (ACE, GLRT-LQ) [E. Conte and M. Lops and G. Ricci, 1995, Kraut and Scharf, 1999]

$$H(\widehat{\Sigma}) = \Lambda_{ANMF}(\mathbf{z}, \widehat{\Sigma}) = \frac{|\mathbf{p}^H \, \widehat{\Sigma}^{-1} \, \mathbf{z}|^2}{(\mathbf{p}^H \, \widehat{\Sigma}^{-1} \, \mathbf{p}) \, (\mathbf{z}^H \, \widehat{\Sigma}^{-1} \, \mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{ANMF} \,, \tag{6}$$

where $\widehat{\Sigma}$ stands for any M-estimators.

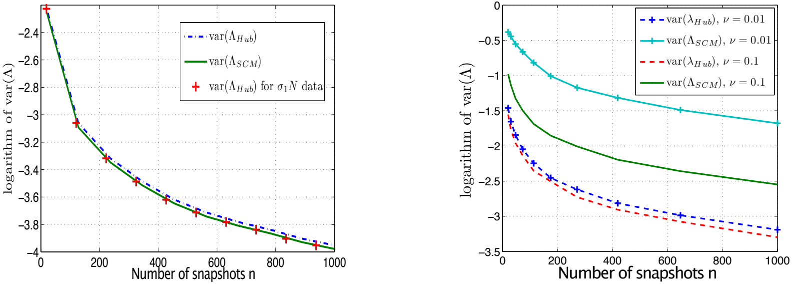
- The ANMF is scale-invariant (homogeneous of degree 0), i.e. $\forall \alpha, \beta \in \mathbb{R}\,,\ \Lambda_{ANMF}(\alpha\, \mathbf{z}, \beta\, \widehat{\Sigma}) = \Lambda_{ANMF}(\mathbf{z}, \widehat{\Sigma})$.
- Its asymptotic distribution (conditionally to $\mathbf{z}$!) is known [Pascal and Ovarlez, 2015, Ovarlez et al., 2015]

$$\sqrt{n}\, \left(H(\widehat{\Sigma}) - H(\Sigma)\right) \xrightarrow{d} \mathcal{CN}\left(0, \sigma_1\, H(\Sigma)\, \left(H(\Sigma) - 1\right)^2\right).$$

- It is CFAR w.r.t the covariance/scatter matrix,
- It is CFAR w.r.t the texture.

Illustrations of the result

- Complex Huber's *M*-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: $\nu = 0.1$ and 0.01).



Validation of theorem (even for small *n*)

Interest of the *M*-estimators

Some comments:

Perfect (but asymptotic) characterization of several objects properties, such as detectors, classifiers, estimators, etc.

$H(SCM)$ and $H(M\text{-estimators})$ share the same asymptotic distribution (differs from σ_1).

$\Downarrow$

- Link to the classical Gaussian case,
- Quantification of the loss involved by robust estimator.

Probability of false alarm

PFA-threshold relation of $\Lambda_{ANMF}(\widehat{\mathrm{S}}_n)$ (Gaussian case, finite n)

$$P_{fa} = (1-\lambda)^{a-1}\,{}_2F_1(a, a-1; b-1; \lambda)\,, \tag{7}$$

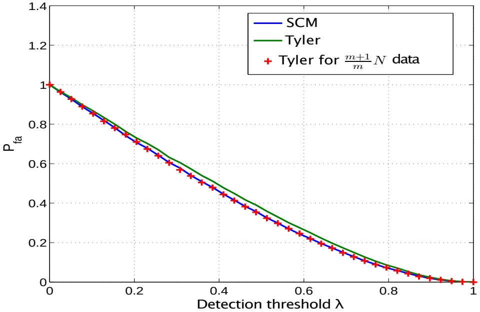
where $a = n - m + 2$, $b = n + 2$ and ${}_2F_1$ is the Hypergeometric function defined as

$$ {}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!}\,. $$

Tyler's estimator: Gaussian context, $n = 10$, $m = 3$

PFA-threshold relation of Λ_{ANMF}(Tyler's est.) for CES distributions

For n large and any elliptically distributed noise, the PFA is still given by (7) if we replace n by $n/\frac{m+1}{m}$.



Probability of false alarm

For n large enough and for any elliptically distributed noise, the PFA is still given by (7) if we replace n by n/σ_1 [Pascal et al., 2004]:

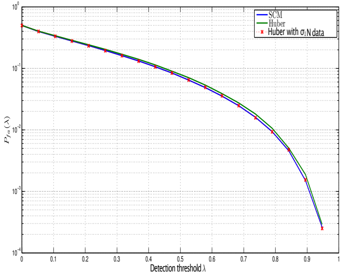
PFA-threshold relation of $\Lambda_{ANMF}(M\text{-est.})$ for CES distributions

$$P_{fa} = (1-\lambda)^{a-1}\, {}_2F_1(a, a-1; b-1; \lambda)\,, \tag{8}$$

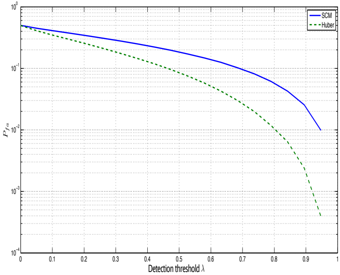
where $a = \dfrac{n}{\sigma_1} - m + 2$, $b = \dfrac{n}{\sigma_1} + 2$ and ${}_2F_1$ is the Hypergeometric function.

Illustrations of the result: Probabilities of False Alarm

- Complex Huber's *M*-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: $\nu = 0.1$).



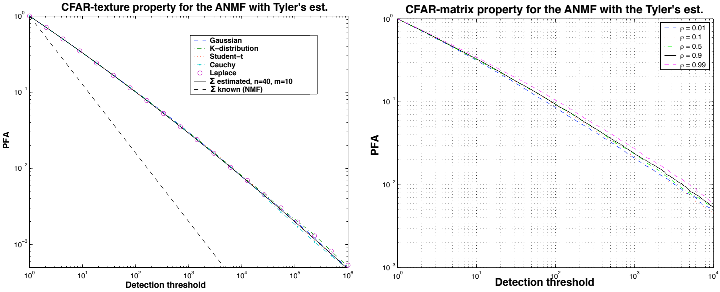
Validation of theorem (even for small *n*)



Interest of the *M*-estimators for False Alarm regulation

Illustration of the ANMF CFAR properties for CES process

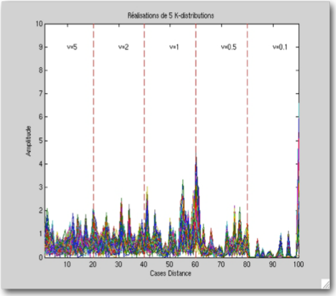
False Alarm regulation for ANMF built with Tyler's estimate

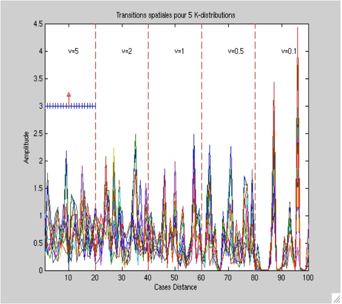


(a) CFAR-texture (b) CFAR-matrix

Figure: Illustration of the CFAR properties of the ANMF built with the Tyler's estimator, for a Toeplitz CM whose (i,j)-entries are $\rho^{|i-j|}$.

Properties of ANMF-Tyler Detector on Clutter Transitions

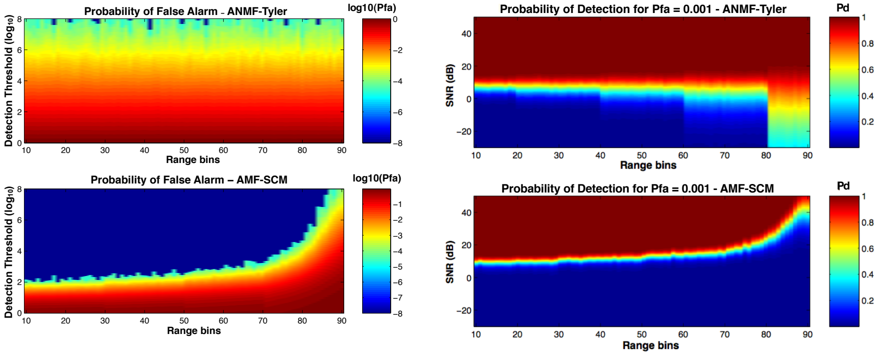




Play/Pause

- K-distributed clutter transitions: from Gaussian to impulsive noise,
- Estimation of the covariance matrix onto a range bins sliding window.

Properties of ANMF-Tyler Detector on Clutter Transitions



- ANMF-Tyler: The same detection threshold is guaranteed for a chosen P_{fa} whatever the clutter area,
- ANMF-Tyler: Performance in terms of detection is kept for moderate non-Gaussian clutter and improved for spiky clutter.

Outline

Robustness of the M-estimators

Let us suppose that $\{\mathbf{y}_i\}_{i=1,n-1} \sim \mathcal{CN}(\mathbf{0}, \Sigma)$ and that the last secondary data $\mathbf{y}_n$ contains outlier $\mathbf{p}_0$:

- Sample Covariance Matrix case:

$$\hat{\mathbf{S}}_n^{pol} = \frac{1}{n}\sum_{k=1}^{n-1} \mathbf{y}_k\,\mathbf{y}_k^H + \frac{1}{n}\mathbf{p}_0\,\mathbf{p}_0^H\,, \qquad E\left[\hat{\mathbf{S}}_n^{pol}\right] = \frac{n-1}{n}\Sigma + \frac{1}{n}E\left[\mathbf{p}_0\,\mathbf{p}_0^H\right].$$

The power of the outlier $\mathbf{p}_0$ has a **big impact** on the quality of the SCM estimation.

- Tyler (or FP) Covariance Matrix case:

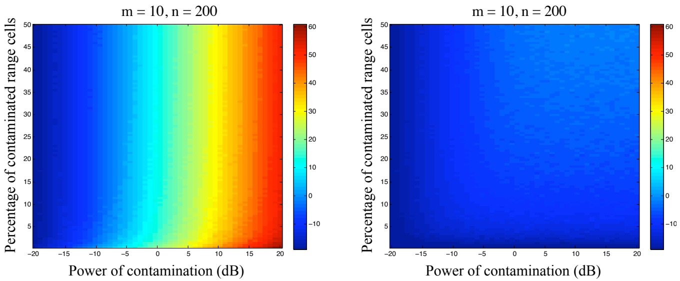
$$\hat{\Sigma}_{FPpol} = \frac{m}{n}\sum_{k=1}^{n} \frac{\mathbf{y}_k\,\mathbf{y}_k^H}{\mathbf{y}_k^H\,\hat{\Sigma}_{FPpol}^{-1}\,\mathbf{y}_k}\,, \qquad E\left[\hat{\Sigma}_{FPpol}\right] = \Sigma + \frac{m+1}{n}\left[E\left[\frac{\mathbf{p}_0\,\mathbf{p}_0^H}{\mathbf{p}_0^H\,\Sigma^{-1}\,\mathbf{p}_0}\right] - \frac{1}{m}\Sigma\right].$$

The power of the outlier $\mathbf{p}_0$ has **no big impact** on the quality of the Tyler estimate.

Robustness of M-estimators

Gaussian vectors $\mathbf{y}_k$ polluted by outliers

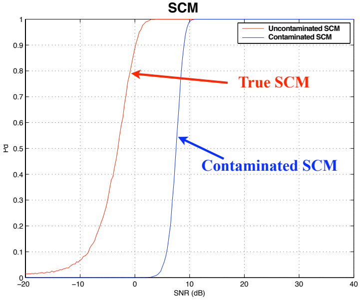
$$\hat{\mathbf{S}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{y}_i\,\mathbf{y}_k^H\,, \qquad \hat{\Sigma}_{FP} = \frac{m}{n}\sum_{k=1}^{n} \frac{\mathbf{y}_k\,\mathbf{y}_k^H}{\mathbf{y}_k^H\,\hat{\Sigma}_{FP}^{-1}\,\mathbf{y}_k}\,.$$



Plot of the error between the covariance matrix estimated with and without ouliers.

Robustness of ANMF: Impact on detection performance

Same target $\mathbf{y}_k = \mathbf{p}_0$ (SNR 20dB) than those in the cell under test in the reference cells (case of convoy for example)



AMF + SCM

ANMF + FPE

The SCM can whiten the target to detect,
The ANMF built with FPE is more robust.

Outline

MUltiple Signal Classification (MUSIC) method

- K (known) direction of arrival θ_k on m antennas
- Gaussian stationary narrowband signal with additive noise.
- the DoA [Schmidt, 1986] is estimated from n snapshots, using the SCM, the Huber's M-estimator and the Tyler's estimator.

$$\mathbf{y}(t) = \mathbf{A}(\theta_0)\,\mathbf{s}(t) + \mathbf{w}(t)\,.$$

- $\theta_0 = (\theta_1, \theta_2, \ldots, \theta_K)^T$,
- the steering matrix $\mathbf{A}(\theta_0) = (\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_K))$,
- $\mathbf{s}(t) = (s_1(t), s_2(t), \ldots, s_K(t))^T$ signal vector,
- $\mathbf{w}(t)$ stationary additive noise.

$$\Sigma = \mathbb{E}[\mathbf{y}\,\mathbf{y}^H] = \mathbf{A}(\theta_0)\,\mathbb{E}[\mathbf{s}\mathbf{s}^H]\,\mathbf{A}^H(\theta_0) + \sigma^2\mathbf{I}\,.$$

which can be rewritten

$$\Sigma = \mathbb{E}[\mathbf{y}\,\mathbf{y}^H] = \mathbf{E}_S\,\mathbf{D}_S\,\mathbf{E}_S^H + \sigma^2\,\mathbf{E}_W\,\mathbf{E}_W^H\,,$$

where $\mathbf{E}_S$ (resp. $\mathbf{E}_W$) are the signal (resp. noise) subspace eigenvectors. The MUSIC statistic is

$$\begin{cases} H(\Sigma) = \underset{\theta}{\operatorname{argmax}}\,\gamma(\theta) & \text{where } \gamma(\theta) = \mathbf{s}(\theta)^H\,\mathbf{E}_W\,\mathbf{E}_W^H\,\mathbf{s}(\theta), \\ H(\widehat{\Sigma}) = \underset{\theta}{\operatorname{argmax}}\,\hat{\gamma}(\theta) & \text{where } \hat{\gamma}(\theta) = \displaystyle\sum_{i=1}^{m-K} \mathbf{s}(\theta)^H\,\hat{\mathbf{e}}_i\,\hat{\mathbf{e}}_i^H\,\mathbf{s}(\theta)\,, \end{cases}$$

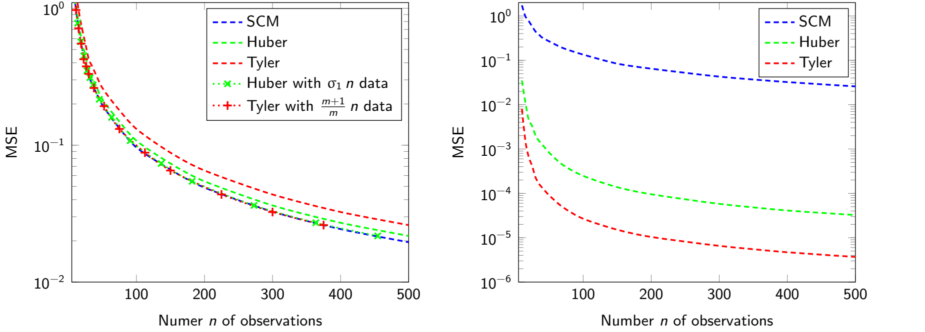
where $\hat{\mathbf{e}}_i$ are the eigenvectors of $\widehat{\Sigma}$.

This function respects assumptions of theorem 2!

Simulation using the MUltiple Signal Classification (MUSIC) method

The Mean Square Error (MSE) between the estimated angle $\widehat{\theta}$ and the real angle θ can then computed (case of one source).

- A $m = 3$ uniform linear array (ULA) with half wavelength sensors spacing is used,
- Gaussian stationary narrowband signal with DoA 20° plus additive noise.
- the DoA is estimated from n snapshots, using the SCM, the Huber's M-estimator and the Tyler's estimator.



(a) White additive Gaussian noise (b) K-distributed additive noise ($\nu = 0.1$)

Figure: MSE of $\hat{\theta}$ vs the number n of observations, with $m = 3$.

Similar conclusions as for detection can be drawn...

Outline

Motivations

The estimation of Σ does not take into account any prior knowledge on the covariance matrix:

> How to improve detection performance by exploiting prior information on Σ ?

$\Longrightarrow$ Use of some prior knowledge on the structure of the covariance matrix:

- Toeplitz: [Burg et al., 1982] for estimation,
- known rank $r < m$ (ex: subspace detector),
- Persymmetry: [Nitzberg and Burke, 1980] for estimation,
  [Cai and Wang, 1992] for detection in Gaussian case,
  [Conte and Maio, 2003, Pailloux et al., 2011] in non-Gaussian noise.

Using Persymmetry Property

Under persymmetric considerations (ex: symmetrically spaced linear array, symmetrically spaced pulse train, ...), the Hermitian covariance matrix Σ verifies: $\Sigma = \mathbf{J}_m \, \Sigma^* \, \mathbf{J}_m$, where $\mathbf{J}_m$ is the m-dimensional antidiagonal matrix having 1 as non-zero elements. If the unitary matrix $\mathbf{T}$ is defined by:

$$\mathbf{T} = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{m/2} & \mathbf{J}_{m/2} \\ i\,\mathbf{I}_{m/2} & -i\,\mathbf{J}_{m/2} \end{pmatrix} & \text{for } m \text{ even} \\ \dfrac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{(m-1)/2} & 0 & \mathbf{J}_{(m-1)/2} \\ 0 & \sqrt{2} & 0 \\ i\,\mathbf{I}_{(m-1)/2} & 0 & -i\,\mathbf{J}_{(m-1)/2} \end{pmatrix} & \text{for } m \text{ odd}, \end{cases} \quad (9)$$

then:

- $\mathbf{s} = \mathbf{T}\,\mathbf{p}$ is a real vector (if $\mathbf{p}$ is centrosymmetric, i.e. $\mathbf{p} = \mathbf{J}_m\,\mathbf{p}^*$),
- $\mathbf{R} = \mathbf{T}\,\Sigma\,\mathbf{T}^H$ is a real symmetric matrix.

Equivalent Detection Problem

Using previous transformation $\mathbf{T}$, the original problem can be reformulated as:

Original Problem	$\mathbf{T}$	**Equivalent Problem**
$\begin{cases} H_0 : \mathbf{y} = \mathbf{c}, & \mathbf{c}_1, \ldots, \mathbf{c}_n \\ H_1 : \mathbf{y} = A\,\mathbf{p} + \mathbf{c}, & \mathbf{c}_1, \ldots, \mathbf{c}_n \end{cases}$	$\rightarrow$	$\begin{cases} H_0 : \mathbf{z} = \mathbf{n}, & \mathbf{n}_1, \ldots, \mathbf{n}_n \\ H_1 : \mathbf{z} = A\,\mathbf{s} + \mathbf{n}, & \mathbf{n}_1, \ldots, \mathbf{n}_n \end{cases}$

where

- $\mathbf{z} = \mathbf{T}\,\mathbf{y} \in \mathbb{C}^m$,
- $\mathbf{n} = \sqrt{\tau}\,\mathbf{x}$ and $\mathbf{n}_k = \sqrt{\tau_k}\,\mathbf{x}_k$ with $\mathbf{x}, \mathbf{x}_k \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{R})$ where $\mathbf{R}$ is an unknown real symmetric matrix,
- $\mathbf{s} = \mathbf{T}\,\mathbf{p}$ is a real vector.

The main motivation for introducing the transformed data is that the original persymmetric complex covariance matrix of the Gaussian speckle Σ is transformed though $\mathbf{T}$ onto a real covariance matrix $\mathbf{R}$.

The Persymmetric FP Covariance Matrix Estimate

From the estimate $\widehat{\mathbf{R}}_{FP}$ of the real covariance matrix $\mathbf{R}$, solution of the following equation:

$$\widehat{\mathbf{R}} = \frac{m}{n} \sum_{k=1}^{n} \frac{\mathbf{n}_k \, \mathbf{n}_k^H}{\mathbf{n}_k^H \, \widehat{\mathbf{R}}^{-1} \, \mathbf{n}_k} \, ,$$

the Persymmetric Fixed-Point Covariance Matrix Estimate can be defined as:

$$\widehat{\mathbf{R}}_{PFP} = \mathcal{R}e(\widehat{\mathbf{R}}_{FP}).$$

Statistical performance of $\widehat{\mathbf{R}}_{PFP}$ [Pailloux et al., 2008, Pailloux et al., 2011]:

- $\widehat{\mathbf{R}}_{PFP}$ is a consistent estimate of $\mathbf{R}$ when n tends to infinity,
- $\widehat{\mathbf{R}}_{PFP}$ is an unbiased estimate of $\mathbf{R}$,
- Its asymptotic distribution is the same as the asymptotic distribution of a real Wishart matrix with $\frac{m}{m+1} 2\, n$ degrees of freedom.

The Persymmetric Adaptive Normalized Matched Filter

The resulting P-ANMF for the transformed problem is based on the PFP estimate and can be defined as:

$$\Lambda(\widehat{\mathbf{R}}_{PFP}) = \frac{|\mathbf{s}^\top \, \widehat{\mathbf{R}}_{PFP}^{-1} \, \mathbf{z}|^2}{(\mathbf{s}^\top \, \widehat{\mathbf{R}}_{PFP}^{-1} \, \mathbf{s})(\mathbf{z}^H \, \widehat{\mathbf{R}}_{PFP}^{-1} \, \mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda. \quad (10)$$

Properties:

- $\Lambda(\widehat{\mathbf{R}}_{PFP})$ is texture-CFAR,
- $\Lambda(\widehat{\mathbf{R}}_{PFP})$ is matrix-CFAR,
- The use of PFP estimate in the ANMF allows to virtually double the number n of secondary data and improve the performance of the ANMF detector built with the FP matrix estimate.

$\Lambda(\widehat{\mathbf{R}}_{PFP})$ is SIRV-CFAR and is called the P-ANMF.

Statistical study of the P-ANMF

The analytical expression for the Probability Density Function of the test statistic $\Lambda(\widehat{\mathbf{R}}_{PFP})$ is really not easy to derive in a closed form but the following results gives some insight about its distribution.

$\Lambda(\widehat{\mathbf{R}}_{PFP})$ has the same distribution as $\dfrac{F}{F+1}$ where

$$F = \frac{(\alpha_1\, u_{22} - \alpha_2\, u_{21})^2 + \left(1 + \left(\dfrac{\beta_3}{u_{33}}\right)^2\right)(a\, u_{22} - b\, u_{21})^2}{(\alpha_2\, u_{11})^2 + \left(t_{11}\, u_{22}\, \dfrac{\beta_3}{u_{33}}\right)^2 + u_{11}^2 \left(1 + \left(\dfrac{\beta_3}{u_{33}}\right)^2\right) b^2} \quad (11)$$

and where: a, b, α_1, $u_{21} \sim \mathcal{N}(0,1)$, $\alpha_2^2 \sim \chi^2_{m-1}$, $\beta_3^2 \sim \chi^2_{m-2}$, $u_{11}^2 \sim \chi^2_{n'-m+1}$, $u_{22}^2 \sim \chi^2_{n'-m+2}$, $u_{33}^2 \sim \chi^2_{n'-m+3}$ with $n' = \dfrac{m}{m+1}\, 2\, n$.

Outline

Conventional Low Rank Detectors

Principle of Low Rank Matched Filter approaches found for example in [Kirsteins and Tufts, 1994] (Principal Component Inverse) and [Haimovich, 1996] (Eigencanceler) and [Rangaswamy et al., 2004].

Let suppose the rank r of clutter covariance matrix Σ is known:

- Example of sidelooking STAP with M pulses measurements and N sensors, $r = N + (M-1)\,\beta$ (Brennan's rule) where $\beta = 2\,v\,T_r/d$.

The idea is to **project the data onto the orthogonal subspace of the clutter**.

$$\hat{\Sigma}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{y}_k\,\mathbf{y}_k^H = (\mathbf{U_r}\,\mathbf{U_0}) \begin{pmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \Sigma_0 \end{pmatrix} (\mathbf{U_r}\,\mathbf{U_0})^H,$$

If we denote by $\mathbf{\Pi}_{SCM} = \mathbf{U}_r\,\mathbf{U}_r^H$ the projector onto the clutter subspace, the Low-Rank ANMF detector is given by:

$$\Lambda_{LR-ANMF-SCM}(\mathbf{z}) = \frac{|\mathbf{p}^H\,(\mathbf{I}-\mathbf{\Pi}_{SCM})\,\mathbf{z}|^2}{(\mathbf{p}^H\,(\mathbf{I}-\mathbf{\Pi}_{SCM})\,\mathbf{p})(\mathbf{z}^H\,(\mathbf{I}-\mathbf{\Pi}_{SCM})\,\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

Extended Low Rank Detectors

In case of heterogeneous and non-Gaussian clutter, we know that $\hat{\Sigma}_{SCM}$ or Π_{SCM} are not good estimates. If we denote the Normalized Sample Covariance Matrix by:

$$\Sigma_{NSCM} = \frac{N\,M}{n} \sum_{k=1}^{n} \frac{\mathbf{y}_k\,\mathbf{y}_k^H}{\mathbf{y}_k^H\,\mathbf{y}_k} = (\mathbf{U_r}\,\mathbf{U_0}) \begin{pmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \Sigma_0 \end{pmatrix} (\mathbf{U_r}\,\mathbf{U_0})^H$$

[Ginolhac et al., 2012, Ginolhac et al., 2013] proved that $\mathbf{\Pi}_{NSCM} = \mathbf{U}_r\,\mathbf{U}_r^H$ is a consistent estimate projector onto the clutter subspace. We can define the extended Low-Rank ANMF-NSCM:

$$\Lambda_{LR-ANMF-NSCM}(\mathbf{y}) = \frac{|\mathbf{p}^H\,(\mathbf{I}-\mathbf{\Pi}_{NSCM})\,\mathbf{z}|^2}{(\mathbf{p}^H\,(\mathbf{I}-\mathbf{\Pi}_{NSCM})\,\mathbf{p})(\mathbf{z}^H\,(\mathbf{I}-\mathbf{\Pi}_{NSCM})\,\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

This detector is found to be **texture-CFAR** and is **asymptotically** Σ**-CFAR**. Moreover, he has another nice **robustness property** when outliers and targets are present in the secondary data. The Normalized Sample Covariance Matrix is a good candidate for adaptive version of Rangaswami's Low Rank Matched Filter and Low Rank Normalized Matched Filter.

More recent works can be found in
[Breloy et al., 2015, Sun et al., 2016, Breloy et al., 2016, Ginolhac and Forster, 2016].

Outline

Shrinkage of Tyler's estimators

Case of small number of observations or under-sampling $n < m$: matrix is not invertible
$\Rightarrow$ Problem when using M-estimators or Tyler's estimator!

Chen estimator

$$\widehat{\Sigma}_C = (1-\beta)\,\frac{m}{n}\sum_{i=1}^{n}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\Sigma}_C^{-1}\mathbf{z}_i} + \beta\mathbf{I}$$

subject to the constraint $\mathrm{Tr}(\widehat{\Sigma}) = m$ and for $\beta \in (0,1]$.

- Originally introduced in [Abramovich and Spencer, 2007],
- Existence, uniqueness and algorithm convergence proved in [Chen et al., 2011],
- Active research [Abramovich and Besson, 2013, Besson and Abramovich, 2013], R. Couillet, M. McKay, A. Wiesel, F. Pascal.

Shrinkage Tyler's estimators

Pascal estimator [Pascal et al., 2014]

$$\widehat{\Sigma}_P = (1-\beta)\,\frac{m}{n}\sum_{i=1}^{n}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\Sigma}_P^{-1}\mathbf{z}_i} + \beta\mathbf{I}$$

subject to the **no** trace constraint but for $\beta \in (\bar{\beta}, 1]$, where $\bar{\beta} := \max(0, 1-n/m)$.

- $\widehat{\Sigma}_P$ (naturally) verifies $\mathrm{Tr}(\widehat{\Sigma}_P^{-1}) = m$ for all $\beta \in (0,1]$,
- Existence, uniqueness and algorithm convergence proved,
- The main challenge is to find the optimal β! [Couillet and McKay, 2014].

Outline

Radar Detection Schemes for Joint Time and Spatial Correlated Clutter

Motivations: Adaptive radar detection and estimation schemes are often based on the independence of the secondary data used for building estimators and detectors. This independence allows to build Likelihood functions.

Example: estimating a covariance matrix $\mathbf{M}$

With a given set of n independent m-dimensional vectors $\{\mathbf{y}_i\}_{i\in[1,n]}$ distributed according to $\mathcal{CN}\left(\mathbf{0}_m, \mathbf{M}\right)$, the corresponding Likelihood function Λ can be built as

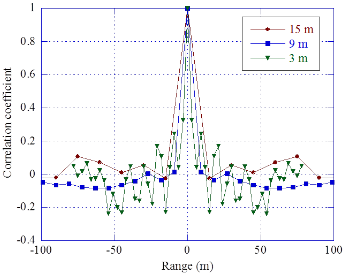
$$\Lambda\left(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \,|\, \mathbf{M}\right) = \prod_{i=1}^{n} p(\mathbf{y}_i) = \prod_{i=1}^{n} \frac{1}{\pi^m \left|\mathbf{M}\right|} \exp\left(-\mathbf{y}_i^H \mathbf{M}^{-1} \mathbf{y}_i\right).$$

The Maximum Likelihood Estimate $\widehat{\mathbf{M}}$ of $\mathbf{M}$ is the zero of the partial derivative of $\Lambda\left(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \,|\, \mathbf{M}\right)$ with respect to $\mathbf{M}$ leading to the well known SCM.

Motivations

In many radar and imagery applications, data $\{\mathbf{y}_i\}_{i\in[1,n]}$ can be viewed as a joint spatial and temporal process:

- For high resolution radar, the sea clutter is clearly jointly spatially and temporally correlated,



Sea clutter spatial correlation, IPIX radar [Greco et al., 2006].

Motivations

- In mutichannel (polarimetric, interferometric or multi-temporal) SAR imaging, the multivariate vector characterizing each spatial pixel of the image is correlated over the channels but can also be strongly correlated with those of neighbourhood pixels,
- When a radar signal with bandwidth B is oversampled ($Fe = k\,B$, $k > 1$), the associated range bins can be spatially correlated and the measurements are not independent anymore.

In the radar community, one generally supposes that the vectors of information collected over a spatial support are identically and independently distributed.

This problem could be, for example, adressed using Multidimensional Space-time ARMA modeling.

The aim of this work is to relax this hypothesis through the use of recent Random Matrix Theory results.

Problem formulation

Detection of a complex signal corrupted by an additive Gaussian noise $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{M})$ in a N-dimensional complex observation vector $\mathbf{y}$:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots, n \\ H_1 : \mathbf{y} = \alpha\, \mathbf{p} + \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots, n \end{cases},$$

where $\mathbf{p}$ is a perfectly known complex steering vector, α is the unknown signal amplitude and where the $\mathbf{c}_i \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{M})$ are n signal-free non independent measurements. The covariance matrix $\mathbf{M}$ characterizes the temporal or spectral correlation within the components of the noise vectors.

To model the spatial dependency between the secondary data, from the Gaussian assumption on $\mathbf{c}_i$, we may write the $m \times n$-matrix $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$ under the following form:

$$\mathbf{C} = \mathbf{M}^{1/2}\, \mathbf{X} \mathbf{T}^{1/2},$$

where $\mathbf{M} \in \mathbb{C}^{m \times m}$ and $\mathbf{T} \in \mathbb{C}^{m \times n}$ are both nonnegative definite, $\mathbf{X}$ is standard Gaussian $\mathcal{CN}(\mathbf{0}_m, \mathbf{I}_m)$, and where $\mathbf{T}$ satisfies the normalization $\dfrac{1}{n}\,\mathrm{tr}(\mathbf{T}) = 1$.

Problem formulation

The matrix $\mathbf{T}$ is considered Toeplitz, i.e., for all i, j, $\mathbf{T}_{i,j} = t_{|i-j|}$ for $t_0 = 1$ and $t_k \in \mathbb{C}$, and positive definite. Besides, $\sum_{k=0}^{n-1} |t_k| < \infty$.

Example: $m = 2$, $n = 3$

$$\mathbf{C} = \underbrace{\underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{1/2}}_{\text{Temporal correlation}} \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}}_{\text{Temporal or Spectral Measurements}} \underbrace{\begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 \end{pmatrix}^{1/2}}_{\text{Spatial correlation}}.$$

Some RMT results

Proposition: Consistent Estimation for $\mathbf{T}$ [Couillet et al., 2015]

As $m, n \to \infty$ such that $m/n \to c \in [0, \infty[$, and for every $\beta < 1$,

$$m^\beta \left\| \mathcal{T}\left[\frac{1}{m}\,\mathbf{C}^H\,\mathbf{C}\right] - \left(\frac{1}{m}\operatorname{tr}\,\mathbf{M}\right)\,\mathbf{T} \right\|_F \xrightarrow{a.s.} 0\,,$$

where $\mathcal{T}[\cdot]$ is the Toeplitzification operator: $(\mathcal{T}[\mathbf{X}])_{ij} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{X}_{k,k+|i-j|}$.

Up to a constant, a consistent estimator $\hat{\mathbf{T}}$ of the spatial covariance $\mathbf{T}$ characterizing data $\{\mathbf{c}_i\}_{i\in[1,n]}$ is therefore defined as $\hat{\mathbf{T}} \propto \mathcal{T}\left[\frac{1}{m}\,\mathbf{C}^H\,\mathbf{C}\right]$ and the associated time whitened sample covariance matrix estimate $\widehat{\mathbf{M}}$ of $\mathbf{M}$ is defined as $\widehat{\mathbf{M}} \propto \frac{1}{n}\,\mathbf{C}\,\hat{\mathbf{T}}^{-1}\,\mathbf{C}^H$.

This technique has been extended in the framework of robust M-estimators.

Gaussian and non-Gaussian scenarios

Simulated Data: joint spatial and time correlated Gaussian or K-distributed ($\nu = 0.5$) data characterized by $m = 10$ pulses, $n = 20$ secondary data where:

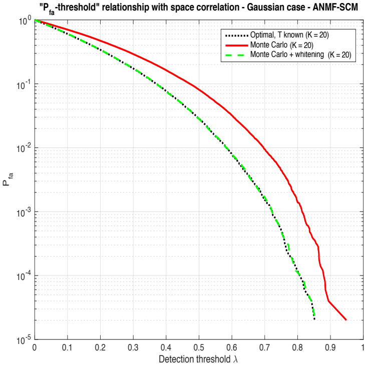
$$\mathbf{M} = \left(\rho_{\mathbf{M}}^{|i-j|}\right)_{i,j \in [1,m]}, \ \mathbf{T} = \left(\rho_{\mathbf{T}}^{|i-j|}\right)_{i,j \in [1,n]} \text{ with } \rho_{\mathbf{M}} = 0.5, \ \rho_{\mathbf{T}} = 0.9.$$

To evaluate the detection performance of the Λ_{ANMF} test statistic, we have compared three approaches:

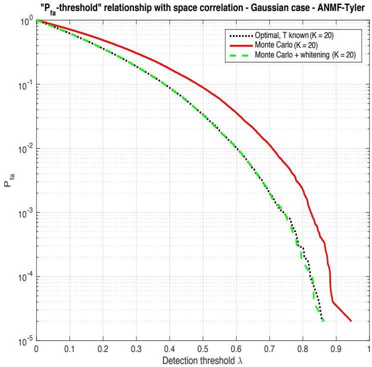
- $\mathbf{M}$ is unknown but $\mathbf{T}$ is assumed to be known: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{n}\mathbf{C}\,\mathbf{T}^{-1}\,\mathbf{C}^H$ (SCM) or the Tyler's estimate of the true spatial-whitened data $\mathbf{C}\,\mathbf{T}^{-1/2}$,
- $\mathbf{T}$ is assumed to be unknown and is estimated through $\hat{\mathbf{T}} \propto \mathcal{T}\left[\frac{1}{m}\mathbf{C}^H\,\mathbf{C}\right]$: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{n}\mathbf{C}\,\hat{\mathbf{T}}^{-1}\,\mathbf{C}^H$ (SCM) or the Tyler's estimate of the spatial-whitened data $\mathbf{C}\,\hat{\mathbf{T}}^{-1/2}$,
- the classical approach that does not take into account the space correlation: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{n}\mathbf{C}\,\mathbf{C}^H$ (SCM) or Tyler's estimate of the data $\mathbf{C}$.

False Alarm Regulation - Gaussian Case

ANMF-SCM



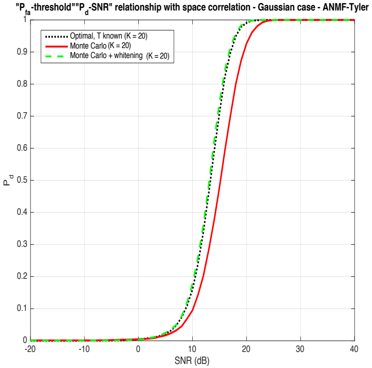
ANMF-Tyler



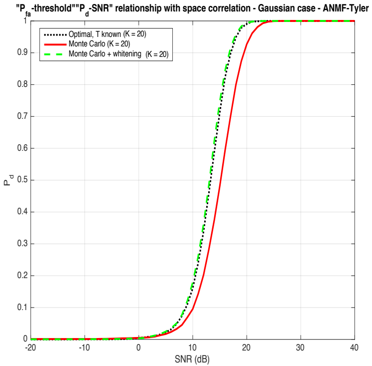
Same False Alarm Regulation performance for ANMF-SCM and ANMF-Tyler (Gaussian case)

Associated Detection Performance - Gaussian Case

ANMF-SCM



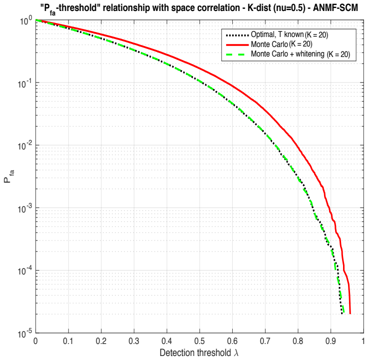
ANMF-Tyler



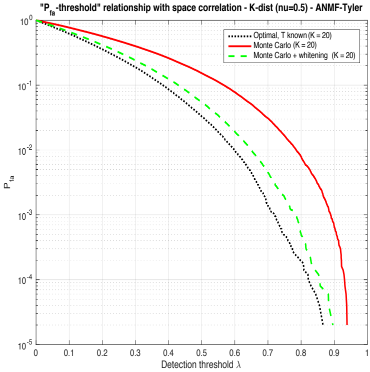
- Same Probability of Detection performance.
- Around 3dB gain improvement with RMT whitening procedure

False Alarm Regulation - K-distributed Case

ANMF-SCM



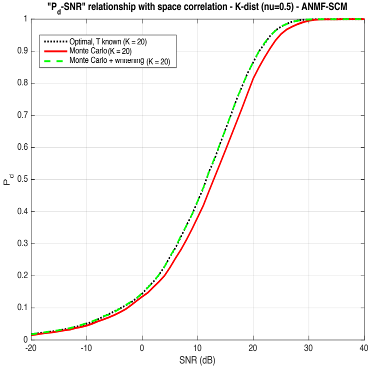
ANMF-Tyler



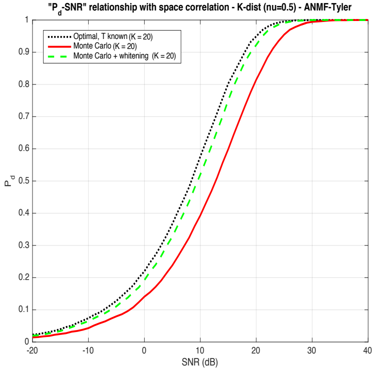
- Better False Alarm regulation performance for ANMF-FP (Non-Gaussian case).
- Better False Alarm regulation with RMT whitening procedure

Associated Detection Performance - K-distributed Case

ANMF-SCM



ANMF-Tyler



- Better performances in terms of Probability of Detection performance for ANMF-Tyler.
  - Around 3dB gain improvement with RMT whitening procedure

End of Part B

Questions?

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