# Structured Matrix Completion: a Convex Relaxation

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**Résumé** – La complétion de données manquantes dans des matrices structurées sous contrainte de rang est un problème d'optimisation non convexe. Une relaxation convexe a été récemment proposée et est basée sur la minimisation de la norme nucléaire (somme des valeurs singulières). Il reste à prouver que ces deux problèmes d'optimisation conduisent bien à la même solution. Dans cette contribution, nous étendons les résultats existants pour des matrices Hankel réelles particulières à des matrices Hankel générales complexes, puis à des matrices quasi-Hankel.

**Abstract** – The completion of structured matrices with missing data under rank constraint is a non convex optimization problem. A convex relaxation has been recently proposed in the case of Hankel matrices, and is based on nuclear norm minimization (the sum of singular values). It remains to prove that the two optimization problems indeed lead to the same solution. In this contribution, existing results on particular real Hankel matrices are extended to general complex Hankel matrices, and then to quasi-Hankel matrices.

**Keywords:** Hankel ; Prony ; tensor CP decomposition ; quasi-Hankel; matrix completion; nuclear norm

## 1 Introduction

The problem of completing matrices with missing entries can be traced back to the works of Prony in 1795, and has been addressed since in various fields including: system identification and control [8, 9, 15], graph theory [17] collaborative filtering [3], compressed sensing [4, 5, 11] information theory [10], chemometrics [2], seismics [13], estimation problems and sensor networks [3], to cite a few. It also appears as a subproblem in the computation of symmetric tensor Canonical Polyadic (CP) decompositions [1].

### 1.1 Matrix completion

We are interested in affine matrix structures (affine maps $\mathbb{C}^N \to \mathbb{C}^{n\times n}$) of the form

$$\mathscr{S}(\boldsymbol{p}) = \boldsymbol{S}_0 + \sum_{k=1}^{N} p_k \boldsymbol{S}_k,$$

where $\boldsymbol{S}_k \in \mathbb{C}^{n\times n}$ are known matrices.

Typically, matrix $\boldsymbol{S}_0$ represents the known part of a matrix, and the Low-Rank Matrix Completion (LRMC) problem consists in finding the vector $\boldsymbol{p}$ so as to minimize the rank of $\mathscr{S}(\boldsymbol{p})$. A convex relaxation of this minimization problem can be obtained by replacing the rank by the nuclear norm (sum of singular values) [16]:

$$\hat{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}\in\mathbb{C}^N} \|\mathscr{S}(\boldsymbol{p})\|_* \tag{1}$$

Our goal is to find when the two minimization problems (*i.e.* rank and nuclear norm) yield the same solution.

Most results in the literature (*e.g.*, [16, 11]), are proved for random structures. To our knowledge, for fixed structure there exists only one result [7], in a very simple case:

**Theorem 1 ([7, Th.1])** *Let $\mathscr{S}$ be the Hankel structure*

$$\mathscr{S}(\boldsymbol{p}) = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^n \\ \lambda & \lambda^2 & \iddots & p_1 \\ \vdots & \iddots & \iddots & \vdots \\ \lambda^n & p_1 & \cdots & p_{n-1} \end{bmatrix},$$

*where $\lambda \in (-1; 1)$. Then the solution of (1) (with constraint $\boldsymbol{p} \in \mathbb{R}^{n-1}$) is unique, is given by $p_k = \lambda^{n+k}$, and coincides with a minimal rank (rank-1) completion.*

In this paper, we extend Theorem 1 in two directions: (i) to arbitrary Hankel complex matrices; and (ii) to quasi-Hankel matrices, which are particularly interesting in the context of symmetric tensor CP decomposition.

### 1.2 Symmetric tensor CP decomposition

Consider a symmetric tensor $\boldsymbol{T}$ of order $d$ and dimension $m$ as an array of numbers with $d$ indices, each varying in

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the range $\{1, \ldots, m\}$. The CP decomposition is:

$$T_{ij..k} = \sum_{r=1}^{R} a_i(r) a_j(r) \,..\, a_k(r) \qquad (2)$$

The minimal number $R$ of terms that are necessary to have an exact fit is called the symmetric tensor rank of $\boldsymbol{T}$.

The CP decomposition is also equivalent to Waring decomposition of a homogeneous polynomial as a sum of powers of linear forms [6]. This equivalence allows to describe the link between (2) and the LRMC for quasi-Hankel matrices [1] (we omit it due to space limitations).

### 1.3 Quasi-Hankel matrices

In the remainder, we shall use multi-indices, which offer a more compact notation [6]. First, for a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{N}^m$, the monomial $x_1^{\alpha_1} x_2^{\alpha_2} .. x_m^{\alpha_m}$ will be denoted as $\boldsymbol{x}^{\boldsymbol{\alpha}}$, and its degree is $|\boldsymbol{\alpha}| = \sum_\ell \alpha_\ell$.

Next, we shall denote by $\blacktriangle^{(m,d)} \subset \mathbb{N}^m$ the set of multi-indices $\{\boldsymbol{\alpha} \in \mathbb{N}^m \colon |\boldsymbol{\alpha}| \le d\}$. For sets $\mathcal{A}, \mathcal{B} \subset \mathbb{N}^m$, we define their Minkowski sum as $\mathcal{A} + \mathcal{B} := \{\boldsymbol{\alpha} + \boldsymbol{\beta} \,|\, \boldsymbol{\alpha} \in \mathcal{A}, \boldsymbol{\beta} \in \mathcal{B}\}$, with a shorthand notation $2\mathcal{A} := \mathcal{A} + \mathcal{A}$. It is easy to see that $\blacktriangle^{(m,d_1)} + \blacktriangle^{(m,d_2)} = \blacktriangle^{(m,d_1+d_2)}$. For $m = 1$, we have that $\blacktriangle^{(1,d)} = \{0, \ldots, d\}$ and $\{0, \ldots, d_1\} + \{0, \ldots, d_2\} = \{0, \ldots, d_1 + d_2\}$. For $m = 2$, an example is shown in Fig. 1 (the multi-indices are depicted as black dots).

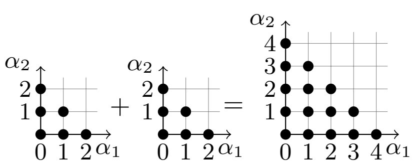


FIG. 1: Minkowski sum of $\blacktriangle^{(2,2)}$ and $\blacktriangle^{(2,2)}$.

Finally, let $\mathcal{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_M\} \subset \mathbb{N}^m$ be an ordered set of multi-indices (according to a certain degree-compatible multi-index order), and $\{h_{\boldsymbol{\alpha}}\}_{\alpha \in 2\mathcal{A}} \subset \mathbb{C}$ be an indexed set of numbers. Then the quasi-Hankel matrix is defined as

$$\mathscr{H}_{\mathcal{A}}(\mathbf{h}) := [h_{\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j}]_{i,j=1}^{M,M}$$

For example, for $\mathcal{A} = \blacktriangle^{(1,d)} = \{0, \ldots, d\}$, $M = d + 1$ and the quasi-Hankel matrix is the ordinary Hankel matrix

$$\mathscr{H}_{\mathcal{A}}(\mathbf{h}) = [h_{k+l}]_{k,l=0}^{d,d} = \begin{bmatrix} h_0 & h_1 & \cdots & h_d \\ h_1 & h_2 & \iddots & h_{d+1} \\ \vdots & \iddots & \iddots & \vdots \\ h_d & h_{d+1} & \cdots & h_{2d} \end{bmatrix}. \qquad (3)$$

For $\mathcal{A} = \blacktriangle^{(2,2)}$ (and $2\mathcal{A} = \blacktriangle^{(2,4)}$, as in Fig. 1), the quasi-Hankel matrix has the form

$$\mathscr{H}_{\mathcal{A}}(\mathbf{h}) = \left[\begin{array}{c|cc|ccc} h_{00} & h_{10} & h_{01} & h_{20} & h_{11} & h_{02} \\ \hline h_{10} & h_{20} & h_{11} & h_{30} & h_{21} & h_{12} \\ h_{01} & h_{11} & h_{02} & h_{21} & h_{12} & h_{03} \\ \hline h_{20} & h_{30} & h_{21} & h_{40} & h_{31} & h_{22} \\ h_{11} & h_{21} & h_{12} & h_{31} & h_{22} & h_{13} \\ h_{02} & h_{12} & h_{03} & h_{22} & h_{13} & h_{04} \end{array}\right]. \qquad (4)$$

### 1.4 The matrix completion problem

The completion problem we consider is the following: Given $\mathcal{A} = \blacktriangle^{(m,d)}$ and the values of $\{h_{\boldsymbol{\alpha}}\}_{\alpha \in \mathcal{A}}$, $h_\alpha \in \mathbb{C}$, we aim at minimizing the rank of $\mathscr{H}_{\mathcal{A}}(\mathbf{h})$ (by optimizing over the remaining elements $\{h_{\boldsymbol{\alpha}}\}_{\alpha \in 2\mathcal{A} \setminus \mathcal{A}}$). For example, in (3), only the values $h_0, \ldots, h_d$ shown in gray are known and $h_{d+1}, \ldots, h_{2d}$ are to be completed. In the general case, the upper block-triangular part of the matrix is known (*e.g.,* in (4) it is shown in gray).

The completion problem can be easily put in the notation of Section 1.1, in terms of $\mathscr{S}(\boldsymbol{p})$: $\boldsymbol{S}_0$ coincides with the matrix $\mathscr{H}_{\mathcal{A}}(\mathbf{h})$ containing all the known elements (and others set to zeros), and $\boldsymbol{S}_k$ are matrices of zeros and ones, with ones put in the positions of unknown elements. For example, for $m = 1$, the matrices $\boldsymbol{S}_k$, $k \ge 1$, have zero elements except on the $(M + k)$th antidiagonal. For an explicit derivation of $\boldsymbol{S}_k$ in the general case see [18].

## 2 Optimality conditions

The main idea of the paper is to consider the cases when the solution to the rank minimization problem is known, and to check that this solution is also a solution of (1).

For this, we use an optimality condition from [18], which is a modified first-order optimality condition suitable for the complex-valued case (a real-valued version of this condition was used in [7]). First, we define the matrix

$$\mathbf{S} := \begin{bmatrix} \operatorname{vec}(\boldsymbol{S}_1) & \ldots & \operatorname{vec}(\boldsymbol{S}_N) \end{bmatrix}, \qquad (5)$$

and for a matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ we define

$$\mathscr{A}(\boldsymbol{P}) := \mathbf{S}^{\mathsf{T}}((\boldsymbol{I} - \boldsymbol{P}) \otimes (\boldsymbol{I} - \boldsymbol{P})) \in \mathbb{R}^{N \times n^2}, \qquad (6)$$

where $\otimes$ denotes the Kronecker product.

**Proposition 2** *Let $\boldsymbol{p}^* \in \mathbb{C}^n$, $\boldsymbol{S}_k$ are real and symmetric, (for $k \in \{1, \ldots, N\}$), and $\mathscr{S}(\boldsymbol{p}^*) = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathsf{H}}$ be an SVD. Then the point $\boldsymbol{p}^*$ is a minimum of (1) iff $\exists \boldsymbol{M} \in \mathbb{C}^{n \times n}$ such that $\|\boldsymbol{M}\|_2 < 1$ and*

$$\mathscr{A}(\boldsymbol{P}) \operatorname{vec}(\boldsymbol{M}) = -\mathbf{S}^{\mathsf{T}} \operatorname{vec}(\boldsymbol{B}) \qquad (7)$$

*is satisfied, where $\boldsymbol{B} := \boldsymbol{U}\boldsymbol{V}^{\mathsf{H}}$, and $\boldsymbol{P} := \boldsymbol{U}\boldsymbol{U}^{\mathsf{H}}$ is the orthogonal projector onto the column space of $\mathscr{S}(\boldsymbol{p}^*)$.*

*If, in addition to (7) it holds that* $\operatorname{rank}\{\mathscr{A}(\boldsymbol{P})\} = N$, *then the point $\boldsymbol{p}^*$ is the unique minimizer of (1).*

It is easy to prove that conditions of the proposition are satisfied for a special class of projectors.

**Lemma 3** *Let $r, s$ be such that $r \le s \le \binom{m + \lfloor \frac{d}{2} \rfloor}{m}$, and $n := \binom{m+d}{m}$. Let $\boldsymbol{P}_0' \in \mathbb{C}^{s \times s}$ be a rank-r projector and*

$$\boldsymbol{P}_0 := \begin{bmatrix} \boldsymbol{P}_0' & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}, \qquad (8)$$

*Then, for the matrices $\boldsymbol{S}_k$ in the quasi-Hankel matrix completion (in Section 1.4), we have that $\boldsymbol{P}_0^{\mathsf{T}} \boldsymbol{S}_k \boldsymbol{P}_0 = \boldsymbol{0}$ for any $k = 1, \ldots, N$. If, in addition, $r \le \binom{m + \lfloor \frac{d-1}{2} \rfloor}{m}$, then* $\operatorname{rank}\{\mathscr{A}(\boldsymbol{P}_0)\} = N$.

A straightforward proof can be found in [18].

# 3 Hankel matrix completion

For Hankel matrices ($m = 1$) the solution of the matrix completion problem is known. We review the solution based on the algebraic theory of Hankel matrices [12].

**Definition 4 ([12])** *Given a finite sequence of complex numbers, $\boldsymbol{h} = [h_0, \dots, h_d]^\mathsf{T}$, the "first characteristic degree" of $\boldsymbol{h}$ (denoted as* hrank$(\boldsymbol{h})$*) is the smallest number $r$ such that $\exists \boldsymbol{q} = [q_0, q_1, \dots, q_{r-1}, q_r]^\mathsf{T} \neq 0$ satisfying:*

$$\boldsymbol{q}^\mathsf{T}[h_k, \dots, h_{k+r}] = 0, \ \forall k \in \{0, \dots, d-r\}.$$

*The corresponding vector $\boldsymbol{q}$ is called the "characteristic vector" of $\boldsymbol{h}$ [12, p.81]. It defines a characteristic polynomial of degree $r$ with $s$ distinct roots:*

$$q(z) = \sum_{j=0}^{r} q_j z^j = c \cdot \sum_{k=1}^{s} (z - \lambda_k)^{\nu_k} \tag{9}$$

*where $\nu_k$ denotes the multiplicity of root $\lambda_k$.*

It is known [12] that for any $\boldsymbol{h}$, hrank$(\boldsymbol{h}) \le \frac{d+2}{2}$; moreover, if hrank$(\boldsymbol{h}) < \frac{d+2}{2}$ then the characteristic vector $\boldsymbol{q}$ is unique (up to scaling). The characteristic polynomial determines the form of $\boldsymbol{h}$. For example, if $q_r \neq 0$ and all the roots $\lambda_k$ are simple, then $\exists c_k$: $h_t = \sum_{k=1}^{r} c_k \lambda_k^t$.

The main result on the completion (see [12]) is:

**Proposition 5** *Let $\boldsymbol{h} \in \mathbb{C}^{d+1}$ be a sequence with a characteristic vector $\boldsymbol{q}$ with $q_r \neq 0$. Then*

- *for the completion* (3)*, the minimal rank is $r =$* hrank$(\boldsymbol{h})$*;*
- *a minimal rank completion is given by the recursion*

$$h_{r+k} = -\frac{1}{q_r} \sum_{j=0}^{r-1} q_j h_{k+j}, \quad \forall k > d - r \tag{10}$$

  *which we will call Canonical Completion. (If $\boldsymbol{q}$ is nonunique, the canonical completion is nonunique);*
- *if $h_t = \sum_{k=1}^{r} c_k \lambda_k^t$, then the minimal rank completion* (10) *is given by the same formula.*

It is easy to see that Theorem 1 treats just the case $\boldsymbol{q} = [-\lambda, 1]^\mathsf{T}$. Next, we consider arbitrary $\boldsymbol{q}$ with $q_r \neq 0$.

**Theorem 6** *For any $d$ and $r < \frac{d+2}{2}$ there exists a constant $\rho = \rho(d, r) > 0$ such that for all $\boldsymbol{h}$ with* hrank$(\boldsymbol{h}) = r$ *having a characteristic vector $\boldsymbol{q}$ with $q_r \neq 0$, and $|\lambda_k| < \rho$, the solution of* (1) *is unique and coincides with the canonical completion* (10).

**Sketch of the proof.**

The main idea is to show that for $\forall \varepsilon > 0$, $\exists \rho > 0$ such that for all corresponding $\boldsymbol{h}$ and with the completion (10), the projector $\boldsymbol{P}$ on the span of $\mathscr{H}_\mathcal{A}(\mathbf{h})$ is close to $\boldsymbol{P}_0 = \left[\begin{smallmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{smallmatrix}\right]$, i.e., $\|\boldsymbol{P} - \boldsymbol{P}_0\|_2 < \epsilon$. The existence of such $\rho$ follows from results on eigenvalues of Toeplitz matrices.

By Lemma 3, we have that $\boldsymbol{P}_0$ satisfies the optimality conditions of Proposition 2. Finally by continuity, the optimality conditions are also satisfied in a neighborhood of $\boldsymbol{P}_0$. The complete proof can be found in [18].

# 4 Quasi-Hankel matrices

Here we consider the general case ($m > 1$), and try to generalize the results of Section 3 to quasi-Hankel matrices. It turns out that some results no longer hold true with the same generality, as subsequently shown.

In Section 4.1 we describe solutions of matrix completion problems for a class of quasi-Hankel matrices. In Section 4.2, we state an analogue of Theorem 6.

## 4.1 Completion

First, consider a class of low-rank quasi-Hankel matrices.

**Lemma 7** *Let $\mathcal{A} = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_M\} \subset \mathbb{N}^m$, and let an array $\{\mathbf{h}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in 2\mathcal{A}}$ be given by $\mathbf{h}_{\boldsymbol{\alpha}} = \sum_{k=1}^{r} c_k \boldsymbol{z}_k^{\boldsymbol{\alpha}}$, for some $c_1, \dots, c_r \in \mathbb{C}$ and $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r \in \mathbb{C}^m$. Then the corresponding quasi-Hankel matrix admits the factorization*

$$\mathscr{H}_\mathcal{A}(\mathbf{h}) = \mathscr{V}_\mathcal{A}(\boldsymbol{z}_1, .., \boldsymbol{z}_r) \operatorname{Diag}\{c_1, .., c_r\} \mathscr{V}_\mathcal{A}^\mathsf{T}(\boldsymbol{z}_1, .., \boldsymbol{z}_r) \tag{11}$$

*where $\mathscr{V}_\mathcal{A}(\boldsymbol{z}_1, .., \boldsymbol{z}_r) := [(\boldsymbol{z}_j)^{\boldsymbol{\alpha}_i}]_{i,j=1}^{M,r}$ is the quasi-Vandermonde matrix.*

For instance, for $\mathcal{A} = \blacktriangle^{(2,2)}$, $r = 3$ and $\boldsymbol{z}_k = \left[\begin{smallmatrix} \lambda_k \\ \mu_k \end{smallmatrix}\right]$, $k = 0, 1, 2$, the quasi-Vandermonde matrix has the form

$$\mathscr{V}_\mathcal{A}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3) = \left[\begin{array}{c|cc|ccc} 1 & \lambda_1 & \mu_1 & \lambda_1^2 & \lambda_1\mu_1 & \mu_1^2 \\ 1 & \lambda_2 & \mu_2 & \lambda_2^2 & \lambda_2\mu_2 & \mu_2^2 \\ 1 & \lambda_3 & \mu_3 & \lambda_3^2 & \lambda_3\mu_3 & \mu_3^2 \end{array}\right]^\mathsf{T}$$

**Definition 8** *Let $\mathcal{A} \subset \mathbb{N}^m$ be a set of multi-indices. We say that the points $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r \in \mathbb{C}^m$ are $\mathcal{A}$-independent if*

$$\operatorname{rank}\{\mathscr{V}_\mathcal{A}(\boldsymbol{z}_1, \dots, \boldsymbol{z}_r)\} = r.$$

The notion of $\mathcal{A}$-independence is equivalent to the fact the monomials $\{\boldsymbol{x}^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{A}}$ taken on the grid of points $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_r\}$ form a set of $\#\mathcal{A} = M$ vectors spanning a linear space of dimension $r$. Hence, these monomials can interpolate any function on this grid. Note also that if $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r$ are $\mathcal{A}$-independent and $c_1, \dots, c_r \in \mathbb{C} \setminus \{0\}$, then $\operatorname{rank}\{\mathscr{H}_\mathcal{A}(\mathbf{h})\} = r$ in Lemma 7.

Finally, assume that the values $\{\mathbf{h}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{A}}$ are known and we have to complete the remaining values $\{\mathbf{h}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in 2\mathcal{A} \setminus \mathcal{A}}$. We describe below the solution based on the flat extension theorem of [14] (see [18] for more details).

**Proposition 9** *Let $\mathcal{A} = \blacktriangle^{(m,d)}$, $d' := \lfloor \frac{d}{2} \rfloor$, $\mathcal{B} := \blacktriangle^{(m,d')}$ (it is easy to se that $2\mathcal{B} \subset \mathcal{A}$). Assume that the values $\{\mathbf{h}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{A}}$ are given as in Lemma 7, where the points $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r$ are $\mathcal{B}$-independent and $c_1, \dots, c_r$ are nonzero. Then, the following hold true*

1. *The rank of the minimal completion in Section 1.4 is $r$. A minimal completion is given by setting*

$$\mathbf{h}_{\boldsymbol{\alpha}} = \sum_{k=1}^{r} c_k \boldsymbol{z}_k^{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in 2\mathcal{A} \setminus \mathcal{A}, \tag{12}$$

   *this will be referred to as the Canonical Completion.*
2. *If $d$ is odd, the completion given in* (12) *is unique.*
3. *If $d$ is even, and $\boldsymbol{z}_1, \dots, \boldsymbol{z}_r$ are $\blacktriangle^{(m,d'-1)}$-independent, then the completion* (12) *is unique.*

### 4.2 Nuclear norm minimization

Now we would like to prove similar results as those of Section 3. First, we show that for quasi-Hankel matrices of the form (11), where points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ are in general position, the limit of certain projectors has the form (8).

**Lemma 10** *Let $\mathcal{A} = \blacktriangle^{(m,d)}$, $r \le \binom{m+d-1}{m}$, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r \in \mathbb{C}^m$ be some points. Furthermore, assume that there exists $0 \le d_0 < d$ such that*

$$\binom{m+d_0-1}{m} =: K < r \le \binom{m+d_0}{m},$$

*and that there exists a set $\mathcal{D}$, $\blacktriangle^{(m,d_0-1)} \subset \mathcal{D} \subseteq \blacktriangle^{(m,d_0)}$, $\#\mathcal{D} = r$ such that the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r$ are $\mathcal{D}$-independent. Let $\boldsymbol{P}(\rho)$ denote the projector onto the column space of $\mathscr{V}_{\mathcal{A}}(\rho\boldsymbol{y}_1, \ldots, \rho\boldsymbol{y}_r)$. Then if $r = \binom{m+d_0}{m}$,*

$$\lim_{\rho\to 0} \boldsymbol{P}(\rho) = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tag{13}$$

*and else if $r < \binom{m+d_0}{m}$,*

$$\lim_{\rho\to 0} \boldsymbol{P}(\rho) = \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tag{14}$$

*where $\boldsymbol{P}_2 \in \mathbb{C}^{L\times L}$ is a projector, with $L = \binom{m+d_0}{m} - K$ and $\operatorname{rank}\{\boldsymbol{P}_2\} = r - K$.*

The proof of Lemma 10 is based on the properties of border bases of polynomial ideals. The main theorem is a consequence of the previous lemmas [18], and the proof is analogous to the proof of Theorem 6.

**Theorem 11** *Let $d'' := \lfloor \frac{d-1}{2} \rfloor$ and $r \le N'' := \binom{m+d''}{m}$. Furthermore, let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r \in \mathbb{C}^m$ satisfy the conditions of Lemma 10. Then there exist a constant $\rho_0 = \rho_0(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r) > 0$ such that for any $\rho$: $0 < \rho < \rho_0$ and points $\boldsymbol{z}_k$ defined as $\boldsymbol{z}_k = \rho\boldsymbol{y}_k$, the following holds true: for any $c_1, \ldots, c_r$ and the initial elements of $\mathbf{h}$ defined in Lemma 7, the canonical completion* (12) *is also the unique solution of* (1).

Note that unlike in Theorem 6, it is not possible to give a universal bound on $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ so that the projector $\boldsymbol{P}$ on the column space of $\mathscr{H}_{\mathcal{A}}(\mathbf{h})$ is arbitrarily close to a $\boldsymbol{P}_0$ as in Lemma 3 (due to fundamental issues in polynomial interpolation). Instead, we showed that for a particular arrangement of points in general position, the points can be rescaled so that $\boldsymbol{P}$ is close to a certain $\boldsymbol{P}_0$. Also, in the case $m = 1$, Theorem 11 is a weak version of Theorem 6.

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