Linear Approximation of the Exponential Map with Application to Simplified Detection in Noncoherent MIMO Systems

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Résumé – Nous étudions une approximation quadratique du Critère de Vraisemblance Généralisé (GLRT) pour pouvoir décoder efficacement une certaine classe de codes espace-temps unitaires pour le canal MIMO à évanouissement par bloc. L'approximation quadratique est dérivée à partir de la série de Taylor de la métrique du GLRT tronqué au deuxième ordre. Une expression analytique du récepteur GLRT approximé est ainsi derivée. Ses performances et les pertes par rapport au cas sans approximation sont illustrées à travers des simulations.

Abstract – We provide a quadratic approximation of the Generalized Likelihood Ratio Test (GLRT) in order to decode efficiently a certain class of unitary space-time codes for the noncoherent MIMO block fading channel. This quadratic approximation is derived from the Taylor series expansion of the GLRT truncated at the second order. A closed form expression of the approximate GLRT receiver in the MIMO case is derived and its performance loss is assessed through some preliminary simulation results.

1 Introduction

Recently, many contributions on constellation design for narrowband noncoherent Multiple Input Multiple Output (MIMO) block fading channels have been proposed in order to exploit their promising diversity and capacity gains. It has been shown that the problem of designing good constellations can be restated as finding optimal packings of M-dimensional subspaces of $\mathbb{C}^T$ ($T > M$), i.e. packings over the Grassmannian $G_{T,M}$, where T is the coherence time of the channel and M it the number of transmit antennas [9].

Here we will focus on the proposal in [6]: starting from a constellation $\mathcal{B}$ carved from a lattice, and by applying to it the so-called exponential map, a constellation $\mathcal{C} \in G_{T,M}$ is obtained: $\mathbf{B} \in \mathcal{B} \xrightarrow{\exp} \mathbf{X} \in \mathcal{C}$. The elements $\mathbf{B} \in \mathcal{B}$ are $(T-M) \times M$ complex matrices; the elements $\mathbf{X} \in \mathcal{C}$ are $T \times M$ complex matrices with unitary orthonormal columns which represent the basis of the corresponding subspaces. While this is an efficient method to generate dense constellations over the Grassmannian, in general $\mathcal{C}$ does not have any apparent structure to be exploited for efficient decoding. However, $\mathcal{B}$ has a lattice structure by construction: its algebraic properties can be of use, if we manage to restate the decoding problem in $\mathcal{B}$. Due to the non-linearity of the exponential map, the decoding metric has in general a complicated expression as a function of $\mathbf{B}$. The idea is then to obtain a local quadratic approximation of the decoding rule, thanks to a Taylor expansion of the decoding metric as a function of $\mathbf{B}$ truncated at the second order.

In this contribution, we generalize to the MIMO case the previous idea, already carried out in the SISO case (one transmit and one receive antenna) in [2]. We derive a local quadratic approximation of the Generalized Likelihood Ratio Test (GLRT) for systems with equal number $M > 1$ of transmit and receive antennas adopting the coding scheme proposed in [6]. We will provide also some performance curve in the case $M = 2$ in order to estimate the performance loss of the approximated decoding rule with respect to the standard one.

2 System Model

Encoder. Let $\mathbf{B}$ be a generic $(T-M) \times M$ complex matrix. The exponential map has the following closed form expression [3]

$$[\mathbf{X}\,\mathbf{X}^{\perp}] = e^{\mathbf{A}} = \exp(\mathbf{A}) = \exp\left(\begin{bmatrix} \mathbf{0} & -\mathbf{B}^{\dagger} \\ \mathbf{B} & \mathbf{0} \end{bmatrix}\right), \qquad (1)$$

where $\mathbf{X}, \mathbf{X}^{\perp}$ are respectively $T \times M$ and $T \times (T-M)$ complex matrices with unitary orthonormal columns, the one being the orthogonal complement of the other: $\mathbf{X}^{\dagger}\mathbf{X}^{\perp} = \mathbf{0}_{M,T-M}$. Let $\mathbf{B} = \mathbf{V}\boldsymbol{\Theta}\mathbf{U}^{\dagger}$ be the thin singular value decomposition (SVD) of $\mathbf{B}$, where $\mathbf{U}$ is $M \times M$ and unitary, $\mathbf{V}$ is $(T-M) \times M$ and has orthonormal columns, $\boldsymbol{\Theta}$ is diagonal and collects the M singular values. The Cosine-Sine (CS) form of (1) is [3]

$$\mathbf{X} = \begin{bmatrix} \mathbf{U}\,\mathbf{C}\,\mathbf{U}^{\dagger} \\ \mathbf{V}\,\mathbf{S}\,\mathbf{U}^{\dagger} \end{bmatrix}, \qquad \mathbf{X}^{\perp} = \begin{bmatrix} \mathbf{U}\,\mathbf{S}\,\mathbf{V}^{\dagger} \\ \mathbf{V}\,\mathbf{C}\,\mathbf{V}^{\dagger} + \mathbf{V}^{\perp}(\mathbf{V}^{\perp})^{\dagger} \end{bmatrix} \qquad (2)$$

where $\mathbf{C} = \cos(\boldsymbol{\Theta})$, $\mathbf{S} = \sin(\boldsymbol{\Theta})$ and $\mathbf{V}^{\perp}$ is any orthogonal complement of $\mathbf{V}$. In order to inverse the exponential map, the following condition on the singular values of the matrices $\mathbf{B} \in \mathcal{B}$ must be satisfied [3]

$$\max_{m=1,\ldots,M} \theta_m(\mathbf{B}_i) < \frac{\pi}{2}, \quad \forall \mathbf{B}_i \in \mathcal{B} \qquad (3)$$

where $\theta_m(\mathbf{B}_i)$ is the m-th singular value of $\mathbf{B}_i$. The procedure to obtain the matrix $\mathbf{B} = \exp^{-1}(\mathbf{X})$ is described in [3]. Given a generic code $\mathcal{B}$, if conditions (3) are not satisfied, a new code is constructed with scaled codewords $\alpha\mathbf{B}_i$, for all i. The common positive scalar α is called *homothetic factor* and modifies only the singular values of $\alpha\mathbf{B}_i$, which become $\alpha\mathbf{\Theta}_i$. The homothetic factor value is chosen to satisfy the inequalities (3) and to optimize the performance of the code, at the same time.

In the following we suppose that $\mathbf{B}$ belongs to a constellation $\mathcal{B}$ carved from a lattice Λ. The matrices $\mathbf{X}$ belong to the corresponding Grassmann code $\mathcal{C}$, generated from $\mathcal{B}$ via the exponential map. In fact, the *sent* codewords are $\mathbf{X} \in \mathcal{C}$. However, the same Grassmann code can be equivalently described also by the set $\mathcal{C}^\perp$ of the matrices $\mathbf{X}^\perp = e^{\mathbf{A}}\,\mathbf{J}$ with $\mathbf{J} = [\mathbf{0}_{T-M,M}\ \ \mathbf{I}_{T-M}]^t$, or by the set $\mathcal{C}_e$ of the matrices $e^{\mathbf{A}}$. In this work, $(\cdot)^t$ stands for simple transposition, while $(\cdot)^\dagger$ denotes transposition and conjugation.

Channel model. The codeword $\mathbf{X} \in \mathcal{C}$ is transmitted through an narrow-band block-fading Rayleigh MIMO channel corrupted by an additive white Gaussian noise (AWGN). T is the length in symbol periods of the coherence length of the channel. The transmitter is equipped with M antennas and the receiver with N antennas: in the following we focus on the case $N = M$. The received complex signal can be written as

$$\underset{T\times M}{\mathbf{Y}} = \underset{T\times M}{\mathbf{X}}\,\underset{M\times M}{\mathbf{H}} + \underset{T\times M}{\mathbf{W}},$$

where each entry of the channel $h_{k,m}$ is a i.i.d. circularly symmetric Gaussian random variable $\mathcal{CN}(0,1)$, whose value is unknown to the transmitter and to the receiver. The entries of the noise matrix are i.i.d. random variables $w_{k,m} \sim \mathcal{CN}(0, \sigma_w^2)$.

Decoder metric. The receiver use a Generalized Likelihood Ratio Test (GLRT) in order to detect the sent matrix symbol. The GLRT can be expressed in many different but equivalent ways, we use the following one

$$\min_{\mathbf{X}^\perp \in \mathcal{C}^\perp} \|\mathbf{Y}^\dagger \mathbf{X}^\perp\|_F^2 = \min_{e^{\mathbf{A}} \in \mathcal{C}_e} \|(\mathbf{J}^t \otimes \mathbf{Y}^\dagger)\,\mathrm{vec}(e^{\mathbf{A}})\|^2\,, \tag{4}$$

where $\|\cdot\|_F$ is the matrix Frobenius norm, and $\mathrm{vec}(\cdot)$ is the operator which concatenates in a unique column vector the columns of the input matrix. The last expression in (4) is obtained by applying the indentity $\mathrm{vec}(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3) = (\mathbf{M}_3^t \otimes \mathbf{M}_1)\mathrm{vec}(\mathbf{M}_2)$, where $\mathbf{M}_i$ are generic complex matrices and $\otimes$ is the Kronecker matrix product. The matrices in (4) can be expressed also in real form: $\mathrm{vec}(\widehat{\mathbf{Y}^\dagger}\widehat{\mathbf{X}^\perp}) = (\hat{\mathbf{J}}^t \otimes \widehat{\mathbf{Y}^\dagger})\,\mathrm{vec}(e^{\hat{\mathbf{A}}})$, where $\hat{(\cdot)}$ is the operator which gives the real form[1] of any complex matrix or vector.

3 Approximation of the GLRT metric

Let us define $\hat{\mathbf{x}} = \mathrm{vec}(e^{\hat{\mathbf{A}}}) \in \hat{\mathcal{C}}_e$, where $\hat{\mathcal{C}}_e$ is exactly equivalent to $\mathcal{C}_e$ but with matrices expressed in real form. Let us define also $\mathbf{P} = \hat{\mathbf{J}}^t \otimes \widehat{\mathbf{Y}^\dagger} = \hat{\mathbf{J}}^t \otimes \hat{\mathbf{Y}}^t$. It is clear that the GLRT metric $q = \|\mathbf{P}\,\hat{\mathbf{x}}\|^2$ is a quadratic function of $\hat{\mathbf{x}}$ but a non-linear function of $\hat{\mathbf{b}} = \widehat{\mathrm{vec}(\mathbf{B})} \in \hat{\mathcal{B}}$ due to the exponential map:

$$q(\hat{\mathbf{b}}) = \hat{\mathbf{x}}(\hat{\mathbf{b}})^t \mathbf{P}^t\,\mathbf{P}\,\hat{\mathbf{x}}(\hat{\mathbf{b}}) = \hat{\mathbf{x}}(\hat{\mathbf{b}})^t \mathbf{Q}\,\hat{\mathbf{x}}(\hat{\mathbf{b}})\,. \tag{5}$$

Unfortunately, $\hat{\mathcal{C}}_e$ has not in general any apparent structure to be efficiently decoded. On the contrary, the corresponding code $\hat{\mathcal{B}}$ has a lattice structure by construction. Hence we can use efficient algorithms to solve closest-point problems (i.e. minimization of a quadratic function) [1]. Following [2], we derive a quadratic approximation of the GLRT metric $q(\hat{\mathbf{b}})$ as a function of $\hat{\mathbf{b}}$, by using its Taylor series truncated at the second order and evaluated in a generic point $\hat{\mathbf{b}}_0$:

$$q(\hat{\mathbf{b}}) \approx q(\hat{\mathbf{b}}_0) + \mathbf{D}_{\hat{\mathbf{b}}}[q](\hat{\mathbf{b}}_0)(\hat{\mathbf{b}} - \hat{\mathbf{b}}_0) \tag{6}$$

$$+ \frac{1}{2}(\hat{\mathbf{b}} - \hat{\mathbf{b}}_0)^t\,\mathbf{D}^2_{\hat{\mathbf{b}}^2}[q](\hat{\mathbf{b}}_0)\,(\hat{\mathbf{b}} - \hat{\mathbf{b}}_0) \tag{7}$$

where $\mathbf{D}_{\hat{\mathbf{b}}}[q](\hat{\mathbf{b}}_0)$ and $\mathbf{D}^2_{\hat{\mathbf{b}}^2}[q](\hat{\mathbf{b}}_0)$ are respectively the first and second differential of the function $q(\hat{\mathbf{b}})$ calculated in $\hat{\mathbf{b}}_0$. Approximation in (6) and (7) is reasonably precise only in a neighborhood of the point $\hat{\mathbf{b}}_0$, in this sense it is just a *local* approximation of the GLRT metric.

The point $\hat{\mathbf{b}}_0$ is obtained from the received signal $\mathbf{Y}$: put $\mathbf{Y}$ in CS form (2) and call it $\mathbf{X}_0$ (see [3] for a procedure to do that). Then $\hat{\mathbf{b}}_0 = \widehat{\mathrm{vec}(\mathbf{B}_0)}$, with $\mathbf{B}_0 = \exp^{-1}(\mathbf{X}_0)$. It holds true that

$$\mathbf{Q}\,\hat{\mathbf{x}}(\hat{\mathbf{b}}_0) = \mathbf{Q}\,\hat{\mathbf{x}}_0 = \mathbf{0}\,, \qquad q(\hat{\mathbf{b}}_0) = 0 \tag{8}$$

since evaluating these expressions comes to calculating $\mathbf{Y}^\dagger\mathbf{X}_0^\perp = \mathbf{0}_{M,T-M}$ by construction of $\hat{\mathbf{b}}_0$.

The first differential in (6) is

$$\mathbf{D}_{\hat{\mathbf{b}}}[q](\hat{\mathbf{b}}_0) = \frac{\mathrm{d}\,q}{\mathrm{d}\,\hat{\mathbf{x}}}(\hat{\mathbf{b}}_0)\,\mathbf{D}_{\hat{\mathbf{b}}}[\hat{\mathbf{x}}](\hat{\mathbf{b}}_0) = 2\hat{\mathbf{x}}_0^t\,\mathbf{Q}\,\mathbf{D}_{\hat{\mathbf{b}}}[\hat{\mathbf{x}}](\hat{\mathbf{b}}_0) = \mathbf{0}\,. \tag{9}$$

where the results are obtained from the differential of symmetric matrices in [7] and introducing (8). Let us define $\mathbf{D}_0 = \mathbf{D}_{\hat{\mathbf{b}}}[\hat{\mathbf{x}}](\hat{\mathbf{b}}_0)$ the first differential of $\hat{\mathbf{x}}$ as a function of $\hat{\mathbf{b}}$ evaluated in $\hat{\mathbf{b}}_0$. The second differential of q as a function of $\hat{\mathbf{b}}$ calculated in $\hat{\mathbf{b}}_0$ is [3]

$$\mathbf{D}^2_{\hat{\mathbf{b}}^2}[q](\hat{\mathbf{b}}_0) = 2\mathbf{D}_0^t\,\mathbf{Q}\,\mathbf{D}_0 + \underbrace{2\big(\mathbf{I}_{4T^2} \otimes (\hat{\mathbf{x}}_0^t\mathbf{Q})\big)\left(\frac{\mathrm{d}^2\,\hat{\mathbf{x}}}{\mathrm{d}\,\hat{\mathbf{b}}^2}(\hat{\mathbf{b}}_0)\right)}_{=\mathbf{0}} \tag{10}$$

where the simplification comes again from (8). Hence, only $\mathbf{D}_0$ is needed, and it is derived by applying formulas in [7] and [8] to our case. By letting $\hat{\mathbf{a}} = \mathrm{vec}(\hat{\mathbf{A}})$ and $\hat{\mathbf{a}}_0$, $\hat{\mathbf{A}}_0$ the corresponding quantities evaluated in $\hat{\mathbf{b}}_0$, it can be shown that the differential takes the following form

$$\mathbf{D}_0 = \left(\frac{e^{\mathbf{K}(\hat{\mathbf{A}}_0^t)} - \mathbf{I}_{4T^2}}{\mathbf{K}(\hat{\mathbf{A}}_0^t)}\right)^t (\mathbf{I}_{2T} \otimes e^{\hat{\mathbf{A}}_0})\,\mathbf{Z}\,. \tag{11}$$

where $\mathbf{K}(\hat{\mathbf{A}}_0) = \hat{\mathbf{A}}_0^t \oplus (-\hat{\mathbf{A}}_0) = (\hat{\mathbf{A}}_0^t \otimes \mathbf{I}_{2T}) - (\mathbf{I}_{2T} \otimes \hat{\mathbf{A}}_0)$. The $(4T^2) \times 2M(T-M)$ real matrix $\mathbf{Z}$ is defined as follows: let $k = 1, \ldots, T-M$ and $\ell = 1, \ldots, M$, its first $M(T-M)$ columns are

$$\mathbf{z}_{k+(T-M)(\ell-1)} = \mathbf{e}_{M+k+2T(\ell-1)} - \mathbf{e}_{\ell+2T(M+k-1)}$$
$$+ \mathbf{e}_{T+M+k+2T(T+\ell-1)} - \mathbf{e}_{T+\ell+2T(T+M+k-1)}$$

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[1] We use the following definition, for any matrix $\mathbf{A}$ or vector $\mathbf{a}$:

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathrm{Re}(\mathbf{A}) & -\mathrm{Im}(\mathbf{A}) \\ \mathrm{Im}(\mathbf{A}) & \mathrm{Re}(\mathbf{A}) \end{bmatrix},\ \hat{\mathbf{a}} = \begin{bmatrix} \mathrm{Re}(\mathbf{a}) \\ \mathrm{Im}(\mathbf{a}) \end{bmatrix}.$$

$\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ are respectively the real and imaginary part. The expression $\widehat{e^{\mathbf{A}}} = e^{\hat{\mathbf{A}}}$ is true, just apply the standard properties of the operator to the series expansion of $e^{\mathbf{A}}$. We express the GLRT metric as a function of real vectors and matrices because results on differentials of real matrices are more widespread in the literature.

and the other $M(T-M)$ columns are

$$\mathbf{z}_{k+(T-M)(M+\ell-1)} = \mathbf{e}_{T+M+k+2T(\ell-1)} - \mathbf{e}_{\ell+2T(T+M+k-1)} + \mathbf{e}_{T+\ell+2T(M+k-1)} - \mathbf{e}_{M+k+2T(T+\ell-1)}$$

where $\mathbf{e}_i$ is the zero column vector of length $4T^2$ with a one at the i-th entry. It can be shown that expression of $\mathbf{D}_0$ in (11) is always well-defined and it can be calculated efficiently.

4 The Simplified Receiver

The expression of the approximate GLRT decoder is

$$\min_{\hat{\mathbf{b}}\in\hat{\mathcal{B}}} \|\mathbf{P}\,\mathbf{D}_0(\hat{\mathbf{b}}-\hat{\mathbf{b}}_0)\|^2 = \min_{\hat{\mathbf{b}}\in\hat{\mathcal{B}}}(\hat{\mathbf{b}}-\hat{\mathbf{b}}_0)^t\tilde{\mathbf{Q}}(\hat{\mathbf{b}}-\hat{\mathbf{b}}_0)\,, \quad (12)$$

with $\mathbf{D}_0$ in (11). The received signal can be always written in the form $\mathbf{Y} = \mathbf{X}_0\mathbf{H}_e$, where $\mathbf{X}_0$ is in CS form $\mathbf{H}_e$ is a sort of channel estimate (if no additive noise is present, then $\mathbf{H}_e = \mathbf{H}$ is the true channel realization and $\mathbf{X}_0$ is the sent codeword). It can be shown that (the derivation is omitted due to lack of space)

$$\mathbf{P}\,\mathbf{D}_0 = (\widehat{\mathbf{J}}^t \otimes \widehat{\mathbf{H}_e}^t\widehat{\mathbf{I}_{T,M}}^t)\mathbf{P}_2\mathbf{Z} = \mathbf{P}_1\mathbf{P}_2\mathbf{Z} \quad (13)$$

where $\mathbf{I}^t_{T,M} = [\mathbf{I}_M\ \ \mathbf{0}_{T-M,M}]$. The matrix $\mathbf{P}_1$ contains explicitly the channel estimate, $\mathbf{Z}$ is a constant matrix and $\mathbf{P}_2$ is a full rank real square matrix which describes the influence of differential of the matrix exponential. In the next subsection we will study in more detail the square symmetric matrix of the quadratic form $\tilde{\mathbf{Q}} = \mathbf{Z}^t\mathbf{P}_2^t\mathbf{P}_1^t\mathbf{P}_1\mathbf{P}_2\mathbf{Z}$.

4.1 The structure of the matrix $\tilde{\mathbf{Q}}$

It is possible to show that the eigenvalues $\lambda_k(\mathbf{P}_2^t\mathbf{P}_2)$ of $\mathbf{P}_2^t\mathbf{P}_2$ are closely related to the singular values $\theta_{m,0} = \theta_m(\mathbf{B}_0)$, $m = 1, \ldots, M$ of $\mathbf{B}_0$ (here we deal with the case $T = 2M$):

$$\lambda_k(\mathbf{P}_2^t\mathbf{P}_2) = 2\frac{1-\cos[\alpha(\theta_{m,0}\pm\theta_{n,0})]}{\alpha^2(\theta_{m,0}\pm\theta_{n,0})}\ ,\ m,n = 1,\ldots,M \quad (14)$$

where we have also shown the homothetic factor α. Since the function $2(1-\cos(x))/x^2$ for $x\in[0,\pi]$ is decreasing from 1 to $4/\pi^2$, we have that $\lambda_{max,2} = \max\{\lambda_k(\mathbf{P}_2^t\mathbf{P}_2)\} = 1$ (obtained when $\theta_{m,0} = \theta_{n,0}$) and $\lambda_{min,2} = \min\{\lambda_k(\mathbf{P}_2^t\mathbf{P}_2)\} = 2(1-\cos(\alpha\delta_{max}))/(\alpha\delta_{max})^2$ (obtained for $\delta_{max} = \max_{m,n}\{\theta_{m,0}+\theta_{n,0}\}$).

By applying the Ostrowski's theorem [4, pag. 224], bounds on the eigenvalues of $\tilde{\mathbf{Q}}$ can be found. In particular

$$\lambda_k(\mathbf{P}_2^t\mathbf{P}_1^t\mathbf{P}_1\mathbf{P}_2) = a_k\lambda_k(\mathbf{P}_1^t\mathbf{P}_1)\,,\ \lambda_{min,2}\le a_k\le 1 \quad (15)$$

where the only non-zero eigenvalues of $\mathbf{P}_1^t\mathbf{P}_1$ are in fact the eigenvalues of $\widehat{\mathbf{H}_e}\widehat{\mathbf{H}_e}^t$ (see definition of $\mathbf{P}_1$ in (13)). Finally, by completing with zeros the matrix $\mathbf{Z}$ to a square matrix $\mathbf{Z}_1 = [\mathbf{Z}\ \mathbf{0}]$, and by noticing that $\mathbf{Z}^t\mathbf{Z} = 4\mathbf{I}_{2(T-M)M}$, we have that $\max\{\lambda_k(\mathbf{Z}_1^t\mathbf{Z}_1)\} = 4$ and $\min\{\lambda_k(\mathbf{Z}_1^t\mathbf{Z}_1)\} = 0$. By applying again the Ostrowski's theorem, and combining with (15), the following bound holds for the non-zero eigenvalues of $\tilde{\mathbf{Q}}$

$$\lambda_k(\tilde{\mathbf{Q}}) = a_kc_k\lambda(\widehat{\mathbf{H}_e}\widehat{\mathbf{H}_e}^t)\quad \begin{array}{l}\lambda_{min,2}\le a_k\le 1\\ 0\le c_k\le 4\end{array}\ . \quad (16)$$

With respect to a coherent system, the structure of the matrix $\tilde{\mathbf{Q}}$ is quite different. Let us suppose that there is no additive noise in the channel. In a coherent system, the channel estimation would be perfect in this case and $\tilde{\mathbf{Q}}$ depends exclusively on $\widehat{\mathbf{H}}\widehat{\mathbf{H}}^t$. On the other hand, in our noncoherent system, even if $\widehat{\mathbf{H}_e} = \widehat{\mathbf{H}}$ when no noise is present, the eigenvalues of $\tilde{\mathbf{Q}}$ are influenced also by the geometry of the code and by the approximation. This is apparent in (15) and in (16), where the coefficients a_k and c_k have possibly vanishing lower bound. The lower bound $\lambda_{min,2}$ over a_k is nonvanishing because $4/\pi^2 < \lambda_{min,2} < 1$ for all codes and all homothetic factors. Still the coefficients a_k attenuate the value of the eigenvalues of $\tilde{\mathbf{Q}}$ with respect to the ones of $\widehat{\mathbf{H}}\widehat{\mathbf{H}}^t$. In general, the higher the homothetic factor, the smaller the lower bound $\lambda_{min,2}$. The coefficients c_k have 0 has lower bound and possibly contribute to a further reduction of the eigenvalues of $\tilde{\mathbf{Q}}$.

The previous remarks, and the fact that the statistics of $\mathbf{B}_0$ are different from the ones of the received signal $\mathbf{Y}$ due to the exponential map non-linearity, let us conclude that the Packet Error Rate (PER) performace of the approximate GLRT will not follow the same law of the one of the GLRT without approximation.

4.2 Simulations

Here we present some simulation results in the case $T = 4$, $M = 2$. We use two coherent codes $\mathbf{B}_1$ and $\mathcal{B}_2$, built as follows

$$\mathcal{B}_1 \ni \mathbf{B} = \frac{1}{\sqrt{2}}\begin{bmatrix} s_1+\phi s_2 & \vartheta(s_3+\phi s_4)\\ \vartheta(s_3-\phi s_4) & s_1-\phi s_2\end{bmatrix}$$

with $\vartheta^2 = \phi = e^{i\pi/4}$ and

$$\mathcal{B}_2 \ni \mathbf{B} = \frac{1}{\sqrt{5}}\begin{bmatrix} \phi_r(s_1+rs_2) & \phi_r(s_3+rs_4)\\ i\bar{\phi}_r(s_3-\bar{r}s_4) & \bar{\phi}_r(s_1+\bar{r}s_2)\end{bmatrix}$$

where $r = (1+\sqrt{5})/2$, $\bar{r} = (1-\sqrt{5})/2$, $\phi_r = 1+i(1-r)$ and $\bar{\phi}_r = 1+i(1-\bar{r})$ (see [3] for more details on these codes). The symbols $s_1,\ s_4,\ s_3,\ s_4$ are selected from a 4-QAM or a 8-QAM alphabeth with unitary energy. Starting from $\mathcal{B}_1$, $\mathcal{B}_2$ we obtain the Grassmann codes $\mathcal{C}_1$, $\mathcal{C}_2$ ($s_k\in$ 4-QAM) or $\mathcal{C}_3$, $\mathcal{C}_4$ ($s_k\in$ 8-QAM). These Grassmann codes and their perfomance depend also on the choice of the homothetic factor α.

In Fig. 1 we present the performace (in PER versus the average SNR per received antenna and per symbol $\mathbf{X}$) for the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ obtained with different homothetic factors. The highest α for each code represent the optimal homothetic factor for that code in the sense that it minimizes the PER performance of the GLRT receiver. In fact, the optimal α guarantees a high minimum distance of the noncoherent code. However, in this case the approximate GLRT decoder is not at all close to the optimal GLRT performance. This is due to the statistics of the proposed receiver (12), which are different from the one of the GLRT. Hence, even if the approximation works and no performance floor is present, the diversity of the system is not recovered yet.

When the homothetic factor decreases, the distance between codewords decreases as well, and the GLRT performance degrades. The approximate GLRT at low SNR it experience a noise-limited channel. In this case, it manages to follow the GLRT performace curve. However, at high SNR, when the

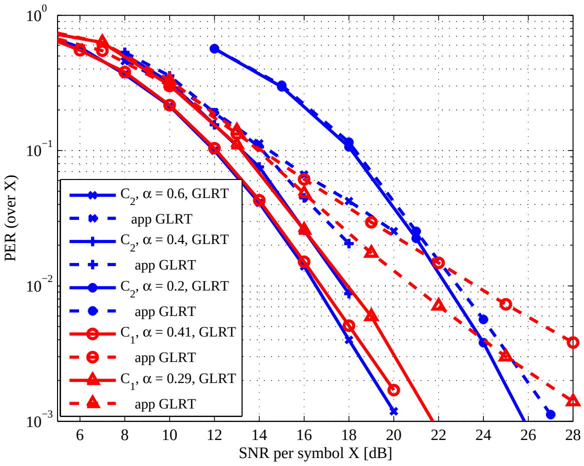

FIG. 1: GLRT (solid lines) and approximate GLRT (dashed lines) performance for codes $\mathcal{C}_1$ and $\mathcal{C}_2$ obtained with different homothetic factors α. Spectral efficiency 2 bit/s/Hz.

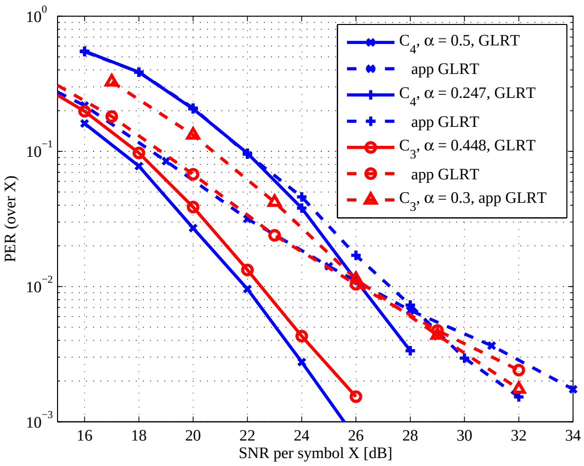

FIG. 2: GLRT (solid lines) and approximate GLRT (dashed lines) performance for codes $\mathcal{C}_3$ and $\mathcal{C}_4$ obtained with different homothetic factors α. Spectral efficiency 3 bit/s/Hz.

noise does no longer dominate, the PER curve of the approximate GLRT is dictated by the asympotic statistics. Then, a loss of diversity is observed.

In Fig. 2 we present the performace for codes $\mathcal{C}_3$ and $\mathcal{C}_4$ with different homothetic factors. Here too, the highest α is the optimal values to minimize the GLRT performance. The behaviour of the approximate GLRT is substantially equivalent to the case of codes $\mathcal{C}_1$ and $\mathcal{C}_2$, even if here we deal with denser constellations.

In this paper we do not directly compare our proposition with other ones, since the aim was to introduce the approximate GLRT decoder and to do some preliminary investigations on its property. However, Jing and Hassibi treated a similar topic was treated [5]. In that work, the authors used another non-linear map (the Cayley map) to generate unitary constellations, and they do some approximation of the GLRT metric in order to decode efficiently. It is not easy to understand if the approximate GLRT receiver in [5] recovers diversity. Moreover, in the case $T = 4$, $M = N = 2$, spectral efficiency equal to 2 bit/s/Hz, the performance proposed in [5, Fig. 2] is comparable to the one obtained in this work (see Fig. 1). However, the simplified receiver proposed in [5] seems to better approximate the true GLRT receiver, even if the optimization of the code is more cumbersome than in our proposal. The reason of the difference in the behaviour of the approximate metric in the two cases is probably to be searched in the fact that in [5] the received signal is preserved, while in our proposal part of it is lost when terms of higher order of the Taylor series are neglected.

5 Conclusion

In this paper we have presented a derivation and first analysis of an approximate GLRT receiver. This receiver applies to unitary constellations suited for transmission on noncoherent MIMO channels, when generated by the exponential map. The approximate GLRT is obtained by truncating the Taylor series expansion of the GLRT metric. This approximation enables efficient decoding of the sent signal, but entails also a loss in the diversity. Results are provided in some cases and a first qualitative comparison with another proposition in the literature is carried out.

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