Asymptotic Outage Capacity of Double Directional MIMO Channels

Mérouane DEBBAH[1]

[1]Forschungszentrum Telekommunikation Wien (FTW)
Vienna, Austria
debbah@ftw.at

Abstract – In this contribution, the double directional model derived within the maximum entropy framework in [1] is studied. An asymptotic analysis (in the number of antennas) is conducted on the achievable transmission limit using tools of random matrix theory. A central limit theorem is provided on the asymptotic behavior of the mutual information and validated in the finite case by simulations. The results are useful in terms of designing a system based on criteria such as quality of service and in optimizing transmissions in multiuser networks.

1 Introduction

The problem of modelling channels[1] is crucial for the efficient design of wireless systems. Unlike the gaussian channel, the wireless channel suffers from constructive /destructive interference signaling [2] which affect the achievable limits of wireless communications. This yields a randomized channel with certain statistics to be discovered. In this part, we introduce the double directional model to be studied. We assume that the transmission takes places between a mobile transmitter and a receiver. The transmitter has t antennas and the receiver has r antennas. Moreover, the input transmitted signal is supposed to go through a time invariant linear filter channel and the interfering noise is additive white gaussian. The transmitted signal and received signal are related as:

$$y(t) = \sqrt{\frac{\rho}{t}} \int \mathbf{H}_{r\times t}(\tau) x(t-\tau) d\tau + n(t)$$

and

$$Y(f) = \sqrt{\frac{\rho}{t}} \mathbf{H}_{r\times t}(f) X(f) + N(f)$$

ρ is the received SNR, $Y(f)$ is the $r \times 1$ received vector (Fourier transform of the time signal $y(t)$), $X(f)$ is the $t \times 1$ transmit vector (Fourier transform of the time signal $x(t)$), $N(f)$ is an $r \times 1$ additive standardized white Gaussian noise vector (Fourier transform of $n(t)$).
In the rest of the paper, we will only be interested in the frequency domain representation. For sake of simplicity, we will write $\mathbf{H}$ instead of $\mathbf{H}(f)$ (without forgetting the dependency on frequency). Recently, in [1], Debbah et al. developed an entropy framework for MIMO channel modelling. The authors derive a consistent[2] double directional model i.e taking into account simultaneously the directions of arrival and the directions of departure. The

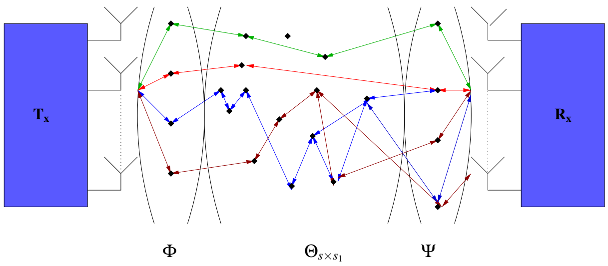

FIG. 1: Double directional based model

motivation of such an approach lies in the fact that when a single bounce on a scatterer occurs, the directions of arrival (DoA) and departure (DoD) are deterministically related by Descartes laws and therefore the distribution of the channel matrix depends on the joint DoA-DoD spectrum. Based on information theoretic tools, the model the authors propose is the following (see figure. (1)):

$$\mathbf{H} = \frac{1}{\sqrt{ss_1}} \begin{pmatrix} e^{j\phi_{1,1}} & \dots & e^{j\phi_{1,s}} \\ \vdots & \ddots & \vdots \\ e^{j\phi_{r,1}} & \dots & e^{j\phi_{r,s}} \end{pmatrix} \begin{pmatrix} P_1^r & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & P_s^r \end{pmatrix}$$
$$\mathbf{\Theta}_{s\times s_1} \begin{pmatrix} P_1^t & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & P_s^t \end{pmatrix} \begin{pmatrix} e^{j\psi_{1,1}} & \dots & e^{j\psi_{1,t}} \\ \vdots & \ddots & \vdots \\ e^{j\psi_{s_1,1}} & \dots & e^{j\psi_{s_1,t}} \end{pmatrix}$$

where $\mathbf{\Theta}_{s\times s_1}$ is an $s \times s_1$ i.i.d zero mean unit variance Gaussian matrix, s and s_1 are respectively the number of DoA scatterers with their respective power P_i^r and the number of DoD scatterers with their respective power P_i^t. $\mathbf{\Theta}_{s\times s_1}$ represents the scattering environment between the set of scatterers s and s_1.

2 Conjecture

Supposing that the model is adequate with reality and that the channel is perfectly known at the receiver, this

[1]This work is part of the European FLOWS (Flexible Convergence of Wireless Standards and Services) project and related documents can be downloaded at http://www.flows-ist.org

[2]this term is defined in their contribution.

contribution will focus on the following conjecture:

Conjecture: Define $I^M(t,r,\rho) = \log_2\det\left(\mathbf{I}_t + \frac{\rho}{t}\mathbf{H}^H\mathbf{H}\right)$ as the mutual information with Gaussian input entries and covariance matrix $\mathbf{Q} = \mathbb{E}(xx^H) = \mathbf{I}$. Then,

$$\lim_{t\to\infty, \frac{r}{t}=\beta} I^M(t,r,\rho) - t\mu \to N(0,\sigma^2) \quad (1)$$

The convergence is in distribution. Only the mean μ and the variance σ^2 are therefore needed to fully characterize the asymptotic distribution. When this conjecture cannot be proved, only the mean will be derived. Note that $\mu = \mu(\beta,\rho)$ and $\sigma^2 = \sigma^2(\beta,\rho)$ depend on $\beta = \frac{r}{t}$ and ρ. Many results have already been derived on the ergodic capacity of channels based on different channel models taking into account correlation [3] or not [4]. However, very few have been devoted to the outage capacity [5]. In this respect, the previous conjecture (1) is extremely useful. Indeed, let q denote the outage probability and $I^M(q)$ the corresponding outage mutual information :

$$\begin{aligned} q &= \int_{-\infty}^{I^M(q)} \mathrm{d}I^M p(I^M) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{I^M(q)} \mathrm{d}I^M e^{-\frac{(I^M - t\mu)^2}{2\sigma^2}} \\ &= 1 - Q\left(\frac{I^M(q) - t\mu}{\sigma}\right) \end{aligned}$$

$$I^M(q) = t\mu + \sigma Q^{-1}(1-q)$$

We define $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \mathrm{d}te^{\frac{-t^2}{2}}$. Therefore, for deriving the outage mutual information, only knowledge of the mean and variance of the mutual information distribution is needed. Hence, for scheduling the network, only information on the mean and the variance needs to be sent to the transmitter. This reduces drastically the overhead of feedback transmissions.

For proving the conjecture, results of the random matrix theory will be used [6]. Random matrices were first proposed by Wigner in quantum mechanics to explain the measured energy levels of nuclei in terms of the eigenvalues of random matrices. One of the useful features of random matrix theory is the ability to predict, under certain conditions, the behavior of the empirical eigenvalue distribution of product or sum of matrices. The results are striking in terms of closeness to simulations with reasonable matrix size and enable us to derive linear spectral statistics for these matrices with only few meaningful parameters.

3 ULA and far field approximation: uncorrelated scattering

We will suppose in this part that $\mathbf{P^r} = \mathbf{I}_r$ and $\mathbf{P^t} = \mathbf{I}_t$. We will take $d = \frac{\lambda}{2}$ and a Uniform Linear Array (ULA). In this case, for example, each column of matrix Φ has a simple expression $[1, e^{-j2\pi\frac{d\sin(\phi)}{\lambda}}, ..., e^{-j2\pi\frac{d(r-1)\sin(\phi)}{\lambda}}]$. We will also suppose that $s \le r$ and $s_1 \le t$. In order to have tractable explicit formulas, we will analyze the distribution of scatterers in the case where for any i there exists a k such as $\sin(\phi_i) = \frac{2k}{r}$ and $\sin(\psi_i) = \frac{2k}{t}$. This case can be seen as an extreme case where all the scatterers are maximally distant from each other (on Fourier directions). Note that this model is similar to Sayeed's framework [7].

Proposition 1 *Let $t \to \infty$, $r \to \infty$, $s \to \infty$, $s_1 \to \infty$ with $\gamma = \frac{r}{s}, \xi = \frac{s}{t}, \gamma_1 = \frac{r}{s_1}, \xi_1 = \frac{s_1}{t}$ then $I^M - t\mu$ converges in distribution to a $N(0,\sigma^2)$ random variable where*[3] *where*

$$\mu = \xi\ln(1 + \rho\gamma - \rho\gamma\alpha) + \xi_1\ln(1 + \rho\gamma_1 - \rho\gamma\alpha) - \xi_1\alpha$$

and

$$\sigma^2 = -\ln\left[1 - \frac{\alpha^2\gamma}{\gamma_1}\right]$$

with

$$\alpha = \frac{1}{2}\left[1 + \frac{\gamma_1}{\gamma} + \frac{1}{\rho\gamma} - \sqrt{(1 + \frac{\gamma_1}{\gamma} + \frac{1}{\rho\gamma})^2 - 4\frac{\gamma_1}{\gamma}}\right]$$

Proof 1 *The proof can be found in [1] and uses a lemma in [6] which deals with linear spectral statistics of the form:*

$$\frac{1}{t}\sum_{i=1}^{t} f(\lambda_i) = \int f(x)dF^{B_t}(x)$$

where $(\lambda_1, ..., \lambda_t)$ denotes the eigenvalues of matrix B_t, $F^{B_t}(\lambda) = \frac{1}{t} \mid \{j : \lambda_j \le \lambda\} \mid$ and f is a function on $[0, \infty[$

Note that in the case where $s = r$ and $s_1 = t$ (which corresponds to an i.i.d Gaussian MIMO channel since the steering matrices are Fourier matrices) then similar formulas as in [5] are obtained. The result is also in accordance with Telatr's asymptotic formulas [4].

In figure 2, simulations have been conducted with $r = t = 8$ antennas and an SNR of 10dB. Three case have been plotted:

- $s = 8$ and $s_1 = 8$
- $s = 4$ and $s_1 = 4$
- $s = 4$ and $s_1 = 8$

In each case, a close match between the theoretical predictions and the simulations occurs. In order to determine the impact of the number of scatterers on the mutual information per receiving antennas, we have plotted in figure (3) the mutual information versus $\xi = \frac{s}{t}$ and $\xi_1 = \frac{s_1}{t}$ for $r = t$. One can observe that due to the fact that $r = t$, the scatterers have the same effect on both the receiving and transmitting side. The maximum rate being of course achieved when $s = s_1 = r = t$.

4 Far field versus Near field

One important question is to know whether, for a given number of scatterers, far field scattering yields better performance than near field scattering. The answer has a direct impact on the understanding and the design of future mobile systems. In the case of far field scattering and

[3] ln is the natural logarithm such $\ln(e) = 1$. When this notation is used, the mutual information is given in nats/s. When the notation $\log_2(x) = \frac{\ln(x)}{\ln(2)}$ is used, the results are given in bits/s.

ULA, the steering directions depend only on the angles. However, in the case of near field scattering, the expression is not simple and depends not only on the angles but also on the position.

In order to have a tractable formula for the near field effect, we will suppose that the angle entries of matrix Φ and Ψ are a realization of independent and uniformly distributed (over $[0, 2\pi]$) variables . The value of the angles do not change during the whole transmission. This is a limiting case of near field scattering (all the rays, for a given scatterer do not come from the same direction)[4]. For comparison purpose, we will also suppose in this part that $\mathbf{P^r} = \mathbf{I}_r$ and $\mathbf{P^t} = \mathbf{I}_t$.

Proposition 2 *Let $t \to \infty$, $r \to \infty$, $s \to \infty$, $s_1 \to \infty$ with $\gamma = \frac{r}{s}, \xi = \frac{s}{t}, \gamma_1 = \frac{r}{s_1}, \xi_1 = \frac{s_1}{t}$ then the mutual information per receiving antenna $\frac{I^M}{t}$ converges to a deterministic value μ where μ is obtained by solving the following equation:*

$$\rho(1-\rho\frac{d\mu}{d\rho})\left[(1-\frac{\rho}{\xi_1}\frac{d\mu}{d\rho})(1-\frac{\rho}{\xi}\frac{d\mu}{d\rho})(1-\frac{\rho}{\gamma\xi}\frac{d\mu}{d\rho})+\frac{1}{\rho}\right] = 1$$

and numerically integrating :

$$\mu = \int \frac{d\mu}{d\rho} d\rho$$

with the boundary condition $\lim_{\rho\to 0}\mu(\rho) = 0$

Proof 2 *The proof can be found in [1] . This result can also be derived within [8] where Müller introduces a N fold MIMO scattering model as a product of N i.i.d random matrices $\mathbf{H} = \prod_{i=1}^{N}\mathbf{M}_i$. He proves the almost sure convergence of the limiting eigenvalue distribution of the matrix $\mathbf{H}$ and gives an explicit form of its Stieljes transform*

We have plotted in figure (4) the theoretical ergodic mutual information per receiving antenna of the near field scenario at 10 dB for various ratio of scatterers s ($\frac{s}{r}$ ranges from 0 to 1) and $r = t$: as a matter of fact, since $r = t$, it does not matter whether one plots the mutual information with respect to $\frac{s}{r}$ or $\frac{s_1}{r}$. s_1 has been chosen to be equal to t. We have also plotted a simulated curve with a system of 8×8 antennas. The angles of arrival were generated randomly according to a uniform distribution and kept fixed during all the trials. A close match between the theoretical formula and the simulations is obtained. We have also plotted the ergodic mutual information of the far field ULA scenario where the scatterers are uncorrelated and given by Fourier directions (see section.3). One can observe that far field scattering (in the case of uncorrelated scatterer) yields better performance than near field scattering. A simple explanation can be provided to this observation: in the far field scenario with uncorrelated scattering and in the case of $s = t$, the DoA matrix Φ and DoD matrix Ψ are unitary Fourier matrices and have therefore no effect on $\mathbf{\Theta}_{s\times s_1}$. However, in the near field scenario, the non-unitary steering matrix Φ and Ψ have a correlation effect on matrix $\Theta_{s\times s_1}$. One of the conclusions of this observation is that a better transmission occurs when the mobile is far from the scatterers and the scatterers are located in distant positions.

[4]We agree on the fact that the near-field case is more complicated as the phases are not completely independent but linked through the geometry of the antenna. We mainly use the random approach in order to have tractable mutual information formulas.

5 Conclusion

In this paper, the asymptotic outage mutual information of a double directional model has been derived and shown to fit simulations with reasonable matrix size. Recently, the results have been extended in order to take into account the power of the steering directions [9]. Moreover, the double directional model as well as its asymptotic analysis have been validated in a recent measurement campaign performed in Oslo [1].

References

[1] M. Debbah and R. Müller, “MIMO Channel Modelling and the Principle of Maximum Entropy: An information Theoretic Point of View ,” submitted to IEEE Transactions on Information Theory, can be downloaded at http://www.flows-ist.org 2003.

[2] E. Biglieri, J. Proakis, and S. Shamai(Shitz), “Fading channels: Information-theoretic and communications aspects,” *IEEE Trans. on Information Theory*, pp. 2619–2692, Oct. 1998.

[3] C. Chuah, D. Tse, J. Kahn, and R. Valenzuela, “Capacity scaling in MIMO wireless systems under correlated fading,” *IEEE Trans. on Information Theory*, pp. 637–650, March 2002.

[4] I.E. Telatar, “Capacity of Multi-antenna Gaussian Channels,” Technical report, AT & T Bell Labs, 1995.

[5] M. Ajith Kamath and B. B. Hughes, “The Asymptotic Capacity of Multiple-Antenna Rayleigh Fading Channels,” *IEEE Trans. on Information Theory*, submitted 2002.

[6] Z.D. Bai and J.W. Silverstein, “Clt of linear spectral statistics of large dimensional sample covariance matrices,” *Accepted for publication in Annals of Probability*, 2003.

[7] A. M. Sayeed, “Deconstructing Multiantenna Fading Channels,” *IEEE Trans. on Signal Processing*, pp. 2563–2579, Oct. 2002.

[8] R. Müller, “On The Asymptotic Eigenvalue Distribution of Concatenated Vector-valued Fading Channels ,” *IEEE Trans. on Information Theory*, pp. 2086 – 2091, Jul 2002.

[9] M. Debbah and R. Müller, “Impact of the power of the steering directions on the capacity of MIMO channels,” in *IEEE International Symposium on Signal Processing and Information Technology*, Darmstadt, Germany, 2003.

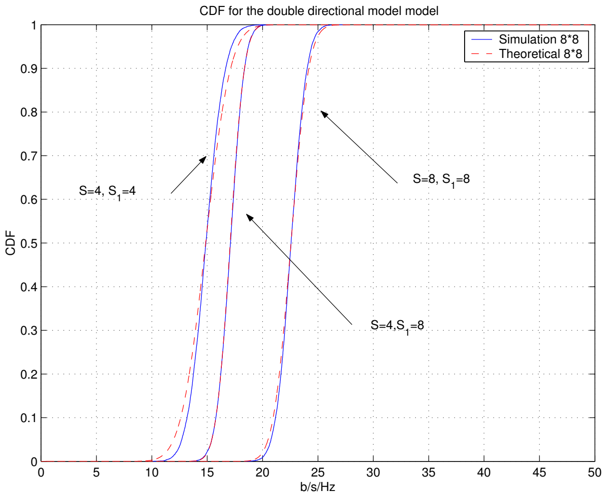

FIG. 2: Mutual information cumulative distribution

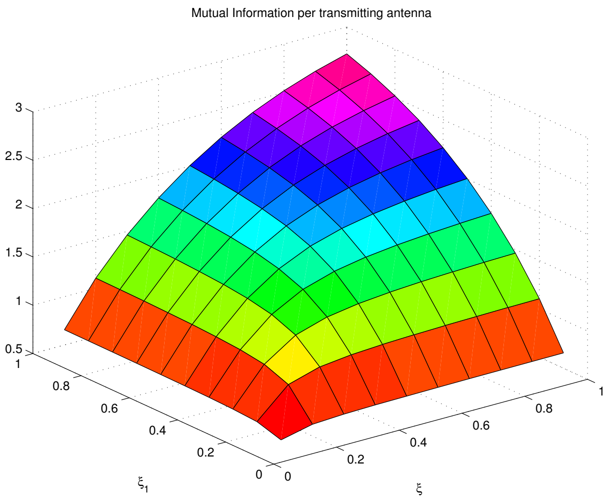

FIG. 3: Mutual Information per transmitting antenna versus ξ and ξ_1

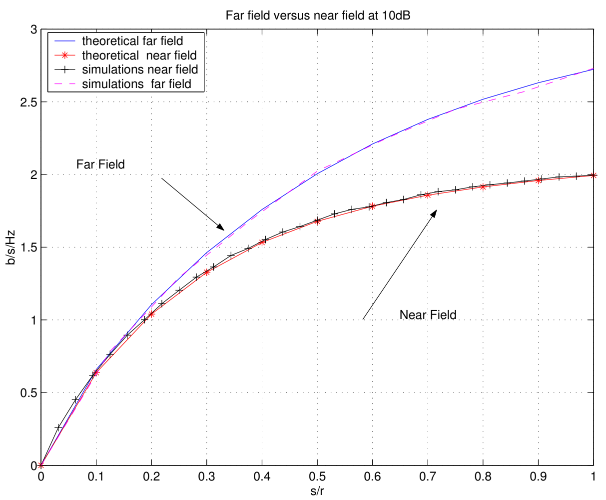

FIG. 4: Far field versus near field scattering at 10dB