Perturbed Sampling in Satellite Images and Reconstruction Algorithms

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Résumé –

Les images satellitaires sont échantillonnées dans une grille qui est légèrement perturbée, à cause des micro-vibrations de l'instrument lors de l'acquisition. Cette perturbation peut être estimée très précisément, mais elle doit aussi être corrigée dans l'image pour certaines applications de stéréoscopie. Dans cette article on montre que les futures satellites en développement au CNES satisfont les conditions requises par la théorie de l'échantillonage irrégulier pour permettre un reéchantillonage stable dans la grille régulière, tandis que la plupart des systèmes imageurs actuels ne le permettent pas à cause d'un niveau trop élevé d'aliasage. A continuation on fait une révision des algorithmes de reconstruction disponibles, et on propose un nouveau algorithme, qui est mieux adapté aux conditions d'échantillonage des futurs satellites. On montre que dans ces conditions l'algorithme proposé est environ deux fois plus rapide que d'autres algorithmes plus généraux.

Abstract –

Satellite images are sampled on a slightly perturbed grid, due to micro-vibrations of the instrument during capture. This perturbation can be estimated with high accuracy, but it must be also corrected in the images for certain stereo and multi-spectral applications. In this work we show how future satellites being developed at CNES satisfy the conditions required by irregular sampling theory to make the problem of resampling on a regular grid well posed, whereas must current imaging systems do not allow for such a well posed reconstruction due to aliasing. Then we discuss the available reconstruction algorithms and propose a new one, which is better adapted to the sampling conditions of such satellites. We show that under such conditions, the proposed algorithm is about twice as fast as other state-of-the-art algorithms.

1 Introduction.

Satellite images are not sampled on an exactly regular grid, but rather on a slightly perturbed grid. The sources of these perturbations include: micro-vibrations of the satellite while it takes the image, and irregularities in the position of the sensors on the image plane. For certain satellite images, the combined effect of these perturbations can be automatically estimated for each image, by different means developed at CNES, and their amplitude is about 0.1 pixels.
Whereas this is a quite small, almost unperceptible perturbation, it must be taken into account by algorithms which interpolate these images to obtain subpixel accuracy. Such an example is the production of highly accurate DEMs from stereo pairs, or superresolution of panchromatic images from multi-spectral images. These applications require image registration with an accuracy even finer than 0.1 pixels in the disparity map. To achieve such an accuracy, the micro-vibrations in the original image sampling must be corrected before registration.

In this work we study the problem of resampling the image on a regular grid, given its samples on perturbed grid and the corresponding perturbation. We note that this perturbation can be obtained with a high level of accuracy from cues given both by gyroscopes mounted on the satellite and by analyzing the images themselves [5]. Nevertheless we shall not deal here with the estimation of the perturbation (*i.e.* the position of the sampling points in the irregular grid), and we shall rather assume that the irregular grid is given with a high level of accuracy.

The article is organized as follows. First (section 2) we study the review the conditions required by irregular sampling theory to make the reconstruction problem well-posed, and we analyse how these conditions apply to satellite imaging systems. Then (section 3) we review some state-of-the-art reconstruction algorithms and point out the characteristics of satellite images that are not exploited by these systems. Next (section 4), we propose a new algorithm, based on a pseudo-inverse iteration, which better exploits these characteristics. Finally, we discuss the results of our simulations (section 5).

2 Existence of a stable reconstruction formula.

Kadec, and later Chui and Shi [8, 1] showed the existence of a stable reconstruction formula for a band-limited function $f \in L^2(\mathbb{R}^2)$ such that $\mathrm{supp}(\hat{f}) \subseteq [-\pi, \pi]^2$, from its samples $\{f(\lambda_k)\}_{k\in\mathbb{Z}^2}$ on an irregular grid, provided that the latter is a small perturbation from the regular grid $\mathbb{Z}^2$, *i.e.* provided that $|\lambda_k - k| < C$ for all $k < \mathbb{Z}^2$. The maximal value of the constant C is unknown in the general two-dimensional case, but the proposition is valid in general for $C = 0.11$, and for certain special kinds of perturbation it is valid for $C = 0.25$ [4].

Most current satellites do not allow such a reconstruction from an irregular sampling grid because they produce highly aliased images[1], thus failing to satisfy the band-limited assumption in Shannon's and Kadec's sampling theorems. The acquisition system of future satellites under study at CNES [6], however, has been designed to satisfy the two hypotheses of Kadec's theorem: on one hand, the transfer function of the instrument is such that the image is almost[2] correctly sampled at the Nyquist rate, and on the other hand the perturbations of the sampling grid are around 0.1 pixels which is below the constant C in Kadec's theorem. This motivates the interest in algorithms for reconstructing the samples $f(k)$ on the regular grid, form the samples $f(\lambda_k)$ on the perturbed grid, and the knowledge of the perturbed grid λ_k.

3 Available Reconstruction Algorithms.

Whereas the demonstration of Kadec's perturbed sampling theorem is constructive, the reconstruction formula it provides is not attractive from a computational point of view.

Even though the algorithmic reconstruction problem has been largely studied in the literature, the work concentrates to a large extent on the one-dimensional irregular sampling problem [2], and it only extends to two dimensions when the perturbation is separable, which is not the general case for satellite images.

Among the algorithms which are valid for a general two dimensional perturbation, the one developed by Gröchenig and Strohmer [3] is the best performing one, to the best of our knowledge, but it assumes a completely *irregular* gridwhereas satellite images: *(i)* are sampled on a more specific *perturbed* grid $\lambda_k = k + \varepsilon(n)$; and *(ii)* the perturbation function ε is not only bounded by a constant smaller than the Kadec constant C, but it is also very smooth with respect to f, *i.e.* $\hat{\varepsilon}$ is usually concentrated in $[-\frac{\pi}{N}, \frac{\pi}{N}]^2$ where $N > 1$ is typically in the order of 10.

4 Pseudo-inverse Algorithm.

For the reasons outlined in the previous section, we developed a the following reconstruction algorithm, which better exploits the sampling conditions that arise in CNES's future satellites.

We note by $y = \Pi_{\mathbb{Z}^2} \cdot f$ the ideal distribution which would result from sampling the original image f on the regular grid. The irregular grid is $\mathbb{Z}^2 + \varepsilon(\mathbb{Z}^2)$ and we note by $\tilde{y} = \Pi_{\mathbb{Z}^2+\varepsilon(\mathbb{Z}^2)} \cdot f$ the distribution that results from sampling the original image f on the perturbed grid. Since f is band-limited (*i.e.* $\mathrm{supp}(\hat{f}) \subseteq [-\pi, \pi]^2$), $\tilde{y}$ can be easily obtained from y by sinc convolution and sampling on the perturbed grid:

$$\tilde{y} = A^+ y = \Pi_{\mathbb{Z}^2+\varepsilon(\mathbb{Z}^2)} \cdot (\mathrm{sinc} * y). \tag{1}$$

In the following we shall abbreviate this convolution-sampling pair by the operator A^+. This is the operation performed by the satellite when it samples on a perturbed grid due to microvibrations. What we look for is to obtain y from $\tilde{y}$, *i.e.* we look for the inverse of A^+, which is still a linear operator, but not a convolution-sampling pair. Nevertheless we can approximate the inverse operator by a convolution-sampling pair (that we shall call A^-) as follows:

$$y^{(1)} = A^- \tilde{y} = \Pi_{\mathbb{Z}^2-\varepsilon(\mathbb{Z}^2)} \cdot (\mathrm{sinc} * \tilde{y}). \tag{2}$$

In the following we shall assume that this is a good approximation to the inverse operator, in the sense that:

$$A^- A^+ = (I - \alpha)\text{ , with } ||\alpha|| < a << 1. \tag{3}$$

We obtained strong experimental evidences for this conjecture, which is the basis for the convergence of our reconstruction algorithm. Note also that in the case of constant ε, A^- is actually the inverse operator and $\alpha = 0$. Since ε is very smooth (locally constant) we should have small values of a.

Under conjecture (3) $y^{(1)}$ in equation (2) gives a good estimate of y, namely $y^{(1)} = (I - \alpha)y$, so the relative approximation error $||y^{(1)} - y||/||y|| < a$ is small. But we can arbitrarily increase the approximation order by the following iteration:

$$y^{(n)} = \alpha y^{(n-1)} + y^{(1)} = (I - \alpha^n)y \tag{4}$$

so the relative error is bounded by a^n at the n-th iteration. In practice the operator α is applied in two steps: First, a simulation $\tilde{y}^{(n-1)}$ of the perturbed image from the current estimate $y^{(n-1)}$ of the regular image:

$$\tilde{y}^{(n-1)} = A^+ y^{(n-1)}; \tag{5}$$

and secondly a correction of the errors found in this simulation with respect to the known perturbed image $\tilde{y}$:

$$y^{(n)} = y^{(n-1)} - A^-(\tilde{y}^{(n-1)} - \tilde{y}). \tag{6}$$

4.1 Numerical Approximation

The pseudo-inverse method described above involves one application of the operators A^+ and A^- per iteration. These operators involve a convolution with the infinite sinc filter, which in practice has to be truncated at some point. Since this convolution is non-separable (because we will sample the convolved image on a non-regular grid), it is important to approximate the sinc filter by a small compactly supported one, which avoids truncation artifacts. Whereas a common practice consists of damping the sinc filter by a Gaussian

$$\mathrm{sinc}(x) \approx \frac{1}{Z} e^{-\frac{x^2}{2\sigma^2}} \mathrm{sinc}(x) \tag{7}$$

and then truncating at say $x = 2\sigma$, a nearly optimal compactly supported approximation is given by the cardinal B-spline [7]:

$$\mathrm{sinc} \approx \beta^{(m)}_{\mathrm{int}} = \mathcal{F}^{-1}\left(\frac{\hat{\beta}^{(m)}}{\sum_{l=-\infty}^{+\infty} \hat{\beta}^{(m)}(w + 2\pi l)}\right) \tag{8}$$

where the m-th order B-spline is defined by the recursion $\beta^{(m)} = \beta^{(0)} * \beta^{(m-1)}$, $\beta^{(0)} = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$. In any case the operator $\alpha = A^- A^+ - I$ in equation (4) will be approximated in our algorithm by $\alpha_\delta = A^-_\delta A^+_\delta - I$, whereas in equation (2) it will be approximated by $\alpha'_\delta = A^-_\delta A^+ - I$, since the first application if A^+ is performed by the acquisition system, not by our algorithm. It is easy to show that $\lim_{\delta\to 0} \alpha_\delta = \lim_{\delta\to 0} \alpha'_\delta =$

[1] About 18% of the image energy may be due to aliasing

[2] By almost correctly sampled we mean that energy of the image that lies outside of the sampling grid's reciprocal cell is below the noise level. In the case of CNES satellites this aliasing represents about 1.5% of the image energy, which is at about the same level as the noise.

α where $\delta = 1/\sigma$ for the damped sinc approximation, and $\delta = 1/m$ for the cardinal B-spline approximation. Hence for a sufficiently small δ we shall have $||\alpha_\delta|| < a < 1$ and $||\alpha'_\delta|| < a < 1$. To summarize, the iterations with the numerical approximation of the sinc filter are as follows:

$$y_\delta^{(1)} = (I - \alpha'_\delta) y \quad (9)$$

$$\begin{aligned} y_\delta^{(n)} &= \alpha_\delta \cdot y_\delta^{(n-1)} + y_\delta^{(1)} \quad (10) \\ &= \left(I + \left(\sum_{i=1}^{n-2} \alpha_\delta^i \right) (\alpha_\delta - \alpha'_\delta) - \alpha_\delta^{n-1} \alpha'_\delta \right) \cdot y \end{aligned}$$

Observe that the relative error $||y_\delta^{(n)} - y|| / ||y||$ still has a term $||\alpha_\delta^{n-1} \alpha'_\delta|| < a^n$ that decreases geometrically with the iterations, but a second term appears which may increase with the iterations but is bounded by $\frac{a}{1-a} ||\alpha_\delta - \alpha'_\delta||$. This second term can be kept below the noise level as long as $a < 1$ and δ is sufficiently small.

5 Experiments.

In order to test the efficiency of the proposed method we constructed a set of simulated satellite images. We started from a very high resolution aerial image g which was later convolved with the transfer function of the satellite to obtain the "analog image" $f = h * g$, which is nearly band-limited, and can be safely sampled at the Nyquist rate to obtain the regular image $y = \Pi_{\mathbb{Z}^2} \cdot f$. Then we used a given perturbation function ε, such that $\mathrm{supp}(\hat{\varepsilon}) \subseteq [-\frac{\pi}{10}, \frac{\pi}{10}]^2$, and $|\varepsilon| < 0.1$, to simulate the perturbed image by a variant of equation (1):

$$\tilde{y} = A_\delta^+ y + \eta. \quad (11)$$

The only differences with equation (1) is that: *(i)* we added a white noise η [3] in order to simulate the level of noise that will be added by the satellite sensors, and *(ii)* the resampling operator A_δ^+ is not infinite, it is rather approximated by a Gaussian-damped sinc with a very large standard deviation $\sigma = 150$ pixels.

Then we used both Gröchening's algorithm and the proposed algorithm with a Gaussian damping ($\sigma = 5$) and a cardinal B-spline (order $m = 11$) approximation of the sinc filter. Figure 1 shows the approximation error $||y^{(n)} - y||$ [4] as a function of the number of floating point operations (flops) per pixel performed by each algorithm. Observe that the proposed algorithm reaches an error level comparable to the noise level after only one iteration (152 flops/pixel with B-splines, or 625 flops/pixel with Gaussian damping), whereas Gröchenig's algorithm requires 420 flops/pixel to achieve the same error level. Do also observe how the error increases later, but stabilizes at a constant level as predicted by equation (10).

An extensive series of tests on several images realistic simulations of future satellite images, as well as synthetic examples, with different transfer functions, noise levels, and perturbation functions, confirmed that the algorithm converged in all cases. In addition in all realistic cases the pseudo-inverse algorithm converged with a number of flops about 2-3 times smaller than

[3] The noise standard deviation is about 0.8 gray-levels, and the energy of η represents about 2% of the energy of y

[4] $||y||$ is about 30

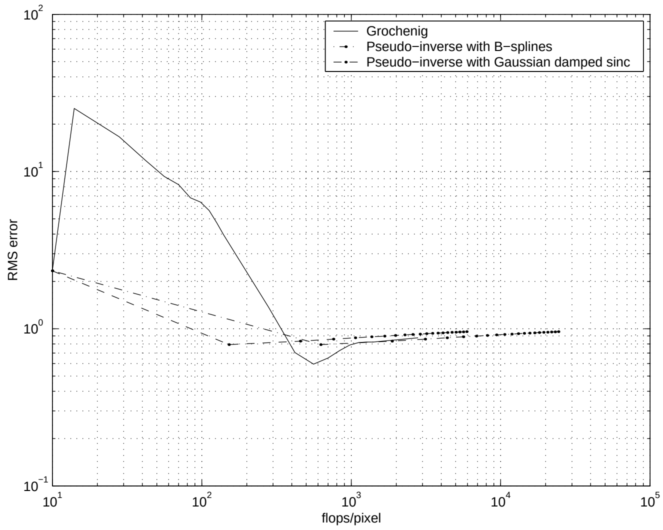

FIG. 1: Approximation error as a function of flops/pixel for the proposed Pseudo-inverse algorithms and Gröchenig's algorithm.

Gröchenig's algorithm. When the noise level is significantly smaller and the spectral contents of the image near the Nyquist frequency is more important (which is not the case for natural images), however, Gröchenig's algorithm converged faster, because in that case the pseudo-inverse algorithm requires more iterations to converge, and each iteration requires a higher order B-spline filter, which takes even more operations per iteration. Nevertheless, even in this case, we could accelerate the performance of the pseudo-inverse algorithm, by starting with low-order B-spline filters and increasing its size as needed during the iterations.

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