A STATE SPACE MODEL FOR NON DIRECTIONAL RANDOM FIELDS

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RÉSUMÉ: Dans la theorie du processement optimal d'antennes on représente généralement les champs aléatoires comme des processus stochastiques à deux indices stationnaires et homogènes. Avec ces représentations, l'antenne processe la fonction de covariance spatial et temporelle du champ réçu ou sa transformée de Fourier, c'est à dire, la fonction de fréquence/numéro d'onde. Pour obtenir des algorithmes récurrents, on modèle les champs réçus comme des sorties des systèmes stochastiques distribués. Dans le cas où on a des champs aléatoires directionels, modèles d'état on été fréquentment usés. Dans cet article, on considère le problème du modelage des champs aléatoires non stationnaires, homogènes et non directionels. Avec l'hypothèse de connaissance de la fonction de covariance espace/temp, on demande que la représentation soit valide sur une antenne linéaire de longueur L. Pour représenter le champ aléatoire non directionel, on recourt à une série de Fourier spatial tronquée. En faisant usage du fait que cette série converge en moyenne quadratique sur la ligne dont la longueur est L, on mesure le degré d'approximation par l'erreur quadratique moyen. Les coefficients de la série sont des processus stochastiques temporels non stationnaires et correlés; ce processus vectoriel peut être interprété comme la sortie d'un système dynamique linéaire, à paramètres variables, avec des entrées stochastiques.

ABSTRACT: Representations of random fields as wide sense stationary and homogeneous stochastic processes are generally used in optimum array processing theory. With these type of representation the array processes the received field space time covariance function or its Fourier transform, the frequency wave number function. In order to get recursive processing algorithms the received fields are modelled as the outputs of distributed stochastic systems. For directional random fields, state space models have been extensively used. In this paper the problem of modelling non stationary but wide sense homogeneous non directional random fields is considered. The space time covariance function is assumed known and the representation needs to be valid only over a line aperture of lenght L. A truncated spatial Fourier series is used to represent the non directional random field. As this series has mean square convergence over the line segment of lenght L, the quality of the approximation can be measured by the mean square error. The coefficients of the truncated series form a vector of correlated non stationary stochastic processes which can be viewed as the output vector of a linear dynamical time varying system with stochastic inputs.

I. INTRODUCTION

In this paper one is concerned with the problem of optimal linear estimation of random fields, i.e., random processes defined over a field with a temporal domain or index set $t \in [T_i, T_f]$ and a spatial domain or index set $\underline{r} \in R^3$. There are several ways to characterize or describe the random fields of interest. Most analysis involve a second moment characterization of the random field $x(t,\underline{r})$ by specifying its mean vector

$$m_x(t,\underline{r}) = E\{x(t,\underline{r})\} \tag{1}$$

and its space/time covariance matrix

$$K_x(t_1,t_2;\underline{r}_1,\underline{r}_2) =$$

$$= E\{[x(t_1,\underline{r}_1)-m_x(t_1,\underline{r}_1)][x(t_2,\underline{r}_2)-m_x(t_2,\underline{r}_2)]^T\}. \tag{2}$$

Here it is assumed that the random fields are Gaussian and so (1) and (2) completely characterize them. For wide sense stationary random fields the mean vector and the space/time covariance matrix can be writen as

$$m_x(t,\underline{r}) = m_x(\underline{r}) \tag{3}$$

and

$$K_x(t_1,t_2;\underline{r}_1,\underline{r}_2) = K_x(t_1-t_2;\underline{r}_1,\underline{r}_2) \ , \tag{4}$$

respectivelly. One also encounters situations in which the random fields are wide sense homogeneous or spatially stationary; for this kind of space/time random processes one can write

$$m_x(t,\underline{r}) = m_x(t) \tag{5}$$

and $$K_x(t_1,t_2;\underline{r}_1,\underline{r}_2) = K_x(t_1,t_2;\underline{r}_1-\underline{r}_2) \ . \tag{6}$$

For the sake of simplicity, it will be assumed that the random fields involved are zero mean. Hence, space/time covariance and correlation matrices are the same. For stationary random fields the temporal frequency, spatial correlation matrix is defined by

$$S_x(w;\underline{r}_1,\underline{r}_2) = \int_{-\infty}^{+\infty} K_x(\tau;\underline{r}_1,\underline{r}_2)\exp(-jw\tau)\, d\tau \ ; \tag{7}$$

clearly, when $\underline{r}_1=\underline{r}_2$ this yields the power spectral density on a specific point in space. Similarly, for an homogeneous random field, one can define the temporal correlation spatial wave number matrix as

$$F_x(t_1,t_2;\underline{k}) =$$

$$= \iiint K_x(t_1,t_2;\underline{r})\exp[j(\underline{k}.\underline{r})]\, d\underline{r} \ , \tag{8}$$

where the integration is performed over the entire spatial domain of the process and $\underline{k}.\underline{r}$ is the inner product between the vector wave number and the positioning vector $\underline{r}$. Finally, for a stationary and homogeneous random field, the frequency wave number matrix is defined as

$$P_x(w;\underline{k}) =$$

$$= \int_{-\infty}^{+\infty} d\tau \iiint d\underline{r}\ K_x(\tau;\underline{r})\exp\{-j[w\tau-(\underline{k}.\underline{r})]\} \ . \tag{9}$$

Another method of analysis involves a description of how the random fields are generated. In order to get recursive linear estimation algorithms the received random fields shall be modelled as the output of linear, possibly distributed, stochastic systems. In section II this problem is discussed for signals propagating in an homogeneous medium and observed over a line aperture of lenght L within an additive white random field. The problem of generating non directional random fields is approached in section III: for homogeneous random fields an approximated model based on a spatial truncated Fourier series over the line segment of lenght L is proposed; the coefficients of the series are correlated, in general non stationary, temporal processes which can be modelled as outputs of a stochastic system. In section IV the linear estimation algorithm presented in section II is extended for the situation when the observation noise has contribution of both white and non directional noises.

II. TIME RECURSIVE OPTIMAL LINEAR FILTERING OF PROPAGATING RANDOM FIELDS

Let $z(t,l)$ be the observation of a propagating vector signal $x(t,l)$ at time instant t and on a point l of a line aperture Ω :

$$z(t,l)=H(t,l)x(t,l)+w(t,l),\quad t\geq T_i,\quad l\in\Omega ,\qquad (10)$$

where $H(.,.)$ is an observation matrix with appropriated dimensions and $w(.,.)$ is a zero mean space/time Gaussian white noise, statistically independent of $x(.,.)$ and with space/time correlation matrix given by

$$E\{w(t_1,l_1)w^T(t_2,l_2)\} =$$
$$= R(t_1,l_1)\,\delta(t_1-t_2)\,\delta(l_1-l_2) ,\qquad (11)$$

$R(.,.)$ being a positive definite matrix. Assuming that the propagation medium is homogeneous, every block element of the vector random field $x(.,.)$ can be written as,

$$x_m(t,l)= x_{om}(t-\tau_m(l|\underline{r}_m)),\quad m=1,2\ldots,M ,\quad t\geq T_i,\quad l\in\Omega ,\qquad (12)$$

where $\tau_m(l|\underline{r}\;)$ is the relative delay to a reference point $l_o\in\Omega$ and $\underline{r}_m$ is the positioning vector of source m. With the condition

$$x(t,l_o)=x_o(t)\qquad (13)$$

one can verify that the random field $x(.,.)$ obeys the partial differential equation (PDE),

$$\left[\frac{\partial}{\partial t} + T(l|\underline{r})\frac{\partial}{\partial l}\right] x(t,l) = 0,\quad t\geq T_i, l\in\Omega .\qquad (14)$$

In this equation, $T(l|\underline{r})$ is a block diagonal matrix with blocks of the form

$$[T(l|\underline{r})]_{mm} = \left[\frac{d\,\tau_m(l|\underline{r}_m)}{d\,l}\right]^{-1} .\; I ,\qquad (15)$$

$\underline{r}$ being a vector formed by all source positioning vectors and I an identity matrix of convenient dimension. The vector process $x_o(.)$ is assumed to be the state of a linear system driven by a white noise process and described by the ordinary differential equation (ODE)

$$\frac{d\,x_o(t)}{d\,t} = F(t)\,x_o(t) + G(t)\,u(t),\quad t\geq T_o,\qquad (16)$$

where $u(.)$ is a zero mean, Gaussian, white noise process with correlation matrix

$$E\{u(t)u^T(\tau)\}= Q(t)\delta(t-\tau) ,\qquad (17)$$

and statistically independent of the Gaussian initial condition $x_o(T_o)$. In order to guarantee the existence of the solution of (14) with (13) for all $t\geq T_i$ and $l\in\Omega$, the initial time instant T_o in (16) shall verify

$$T_o \leq T_i - \max_{\substack{m=1,\ldots M\\ l\in\Omega}} \{\tau_m(l)\} .$$

Here one notes that this model is a quite general one because a block element, say $x_{om}(.)$, of the vector random field $x(.,.)$ can be viewed, not only as an uncorrelated interference to any other element of $x(.,.)$ (uncoherent sources), but also as a correlated version of some other elements of $x(.,.)$ (multipath situations).

The problem now is to find the optimal estimate $\hat{x}(t,l|Z_{t,\Omega})$ as a linear combination of all the data

$$Z_{t,\Omega}= \{z(\tau,\lambda),\; T_i\leq \tau \leq t,\; \lambda\in\Omega\}$$

and such that minimizes the mean square error (MSE)

$$tr(E\{[x(t,l)-\hat{x}(t,l|Z_{t,\Omega})]\,[x(t,l)-\hat{x}(t,l|Z_{t,\Omega})]^T\}) .$$

Using the innovations approach ([1],[2]), it is trivial to show that the MMSE filter is governed by the PDE

$$\left|\frac{\partial}{\partial t} + T(l|\underline{r})\frac{\partial}{\partial l}\right| \hat{x}(t,l|Z_{t,\Omega}) =$$
$$= \int_\Omega P(t;l,\sigma)\,H^T(t,\sigma)\,R^{-1}(t,\sigma)\;\nu(t,\sigma)\,d\sigma ,\quad t\geq T_i,\quad l\in\Omega ,\qquad (18)$$

with the condition in $l_o\in\Omega$ given by the ODE

$$\frac{d\,\hat{x}_o(t|Z_{t,\Omega})}{d\,t} = F(t)\,\hat{x}_o(t|Z_{t,\Omega}) +$$
$$+ \int_\Omega P(t;l_o,\sigma)\,H^T(t,\sigma)\,R^{-1}(t,\sigma)\;\nu(t,\sigma)\,d\sigma ,\quad t\geq T_i ,\qquad (19)$$

where the innovations field $\nu(.,.)$, equivalent to the observation process $z(.,.)$, is defined as

$$\nu(t,l) = z(t,l)-H(t,l)\,\hat{x}(t,l|Z_{t,\Omega}) ,\quad t\geq T_i ,\; l\in\Omega .\qquad (20)$$

It can be shown that the error covariance matrix $P(t;l,\lambda)$ is the solution of the following Riccati PDE

$$\frac{\partial P(t;l,\lambda)}{\partial t} + T(l|\underline{r})\frac{\partial P(t;l,\lambda)}{\partial l} + \frac{\partial P(t;l,\lambda)}{\partial\lambda}\,T^T(\lambda|\underline{r}) =$$
$$= -\int_\Omega P(t;l,\sigma)H^T(t,\sigma)R^{-1}(t,\sigma)H(t,\sigma)P(t;\sigma,\lambda)\,d\sigma ,\quad t\geq T_i,\; l,\lambda\in\Omega\; ;(21\text{-a})$$

with spatial conditions in $l_o\in\Omega$ given by

$$\frac{\partial P(t;l_o,l)}{\partial t} + \frac{\partial P(t;l_o,l)}{\partial l}\,T^T(l|\underline{r}) =$$
$$= P(t;l_o,l)F^T(t) -$$
$$- \int_\Omega P(t;l_o,\sigma)\,H^T(t,\sigma)R^{-1}(t,\sigma)H(t,\sigma)P(t;\sigma,l)\,d\sigma ,\quad t\geq T_i,\quad l\in\Omega\; ;\qquad (21\text{-b})$$

$$P(t;l,l_o)=P^T(t;l_o,l), \quad t \geq T_i, \quad l \varepsilon \Omega \; ; \qquad (21\text{-c})$$

$$\frac{dP(t;l_o,l_o)}{dt} = F(t)P(t;l_o,l_o)+P(t;l_o,l_o)F^T(t)+ + G(t)Q(t)G^T(t) - - \int_\Omega P(t;l_o,\sigma)\, H^T(t,\sigma)R^{-1}(t,\sigma)H(t,\sigma)P(t;\sigma,l_o)\, d\sigma \,, \quad t \geq T_i. \qquad (21\text{-d})$$

At this point one notices that the observation field (10) can be viewed as the functionally delayed output of the stochastic linear system governed by the ODE (16) and so (18), (19) and (21) shall be interpreted as natural generalizations of Kwakernaak's filter equations [3].

III. GENERATION OF NON DIRECTIONAL HOMOGENEOUS RANDOM FIELDS

Baggeroer [4] proposed an approach for generating the temporal frequency spatial correlation function associated with ambient non directional noise fields, e.g., the isotropic or omnidirectional noise. At a temporal frequency w_o, those fields, assumed to be wide sense stationary and homogeneous, are modelled as a superposition of uncorrelated infinitesimal plane wave processes, all radiating towards a common point and generated on the surface of a sphere with a large radius when compared with any geometries and wavelenghts of interest. Using spherical coordinates, the temporal frequency correlation function of a non directional noise field is given by

$$S_y(w_o;\underline{r}) = = \int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta}{4\pi}\, S_o(w_o;\theta,\Phi)\, \exp[-jk_o(\underline{a}_r(\theta,\Phi).\underline{r})]\, d\theta\, d\Phi$$

where $\underline{a}_r$ is a unit vector in the radial direction and k_o is related with the wavelenght λ_o by the formulas $k_o=2\pi/\lambda_o=w_o/c$, being c the propagation velocity. If the statistical level is assumed to be uniform for all directions, i.e., $S_o(w_o,\theta,\Phi)=S_o(w_o)$ for $(\theta,\Phi)\varepsilon[0,\pi]\times[0,2\pi]$, then the temporal frequency spatial correlation function of isotropic noise is achieved:

$$S_y(w_o,\underline{r}) = S_o(w_o)\, \mathrm{sinc}(k_o|\underline{r}|) \,,$$

the sinc(.) and |.| functions being defined as sin(.)/(.) and as the Euclidean norm respectivelly. In the last equation, notice the factoring of temporal and spatial structures. By Fourier transforming, one gets the associated frequency wave number function

$$P_y(w_o,\underline{k}) = \begin{cases} \lambda_o/2\; S_o(w_o), & |\underline{k}|\leq k_o \\ 0 & , \text{ elsewhere} \end{cases}$$

which is a band limited function in the wave number space. These properties are common to any non directional random fields with a propagating structure. To generalize this representation of stationary and homogeneous non directional random fields, one considers non stationary background space time noise processes with known space time covariance functions of the form,

$$K_y(t_1,t_2;\underline{r}_1,\underline{r}_2) = K_o(t_1,t_2)\, K_s(\underline{r}_1-\underline{r}_2) \,. \qquad (22)$$

Further, it is assumed that a factorization exists for the covariance function $K_o(t_1,t_2)$; hence, the associated scalar time process $y_o(t)$ has a minimal degree state space representation [5] of the form:

$$\frac{dy(t)}{dt} = F'(t)\, y(t) + G'(t)\, u'(t) \,, \qquad (23\text{-a})$$

$$y_o(t) = H'(t)\, y(t), \qquad t \geq T_i \,, \qquad (23\text{-b})$$

with $u'(t)$ a white noise vector with covariance matrix $Q'(t)\delta(t-\tau)$. As it has been assumed, the observation region is a line aperture of lenght L. So, the representation of the non directional field $y(t;\underline{r})$ needs to be valid only over the space interval $\Omega=[-L/2\leq l\leq L/2]$. It is well known that second order (finite mean square value) wide sense stationary temporal processes have a Fourier series expansion in a time interval T which, for every $t\,\varepsilon\,T$, converges in the mean square sense to the true process. Generalizing this concept to second order wide sense homogeneous space time processes, one can write

$$y_M(t,l) = \sum_{m=-M}^{M} y_{om}(t)\, \exp(-jmk_L l) \,, \qquad l\varepsilon\Omega \,, \quad (24)$$

where the coefficients are temporal processes given by

$$y_{om}(t) = (1/L) \int_\Omega y(t,l)\, \exp(jmk_L l)\, dl \,, \qquad (25)$$

with

$$k_L = 2\pi / L \,. \qquad (26)$$

It can be shown [6] that the MSE

$$E\{|\tilde{y}_M(t,l)|^2\} = E\{|y(t,l)-y_M(t,l)|^2\},$$

converges to zero when M approaches infinity, i.e.,

$$\underset{M\to\infty}{\text{l.i.m.}}\; y_M(t,l) = y(t,l), \qquad l\varepsilon\Omega \,.$$

The temporal processes given by (25) have crosscorrelation functions given by

$$K_{o_{mn}}(t_1,t_2) = = K_o(t_1,t_2)\, \{\cos[(m-n)\pi] \int_{-\infty}^{+\infty} F_s(k)\, \mathrm{sinc}[(k-nk_L)L/2]\, \mathrm{sinc}[(k-mk_L)L/2]\, dk\,\} \,, \quad (27)$$

where $F_s(k)$ is the spacial Fourier transform of $K_s(l)$.

As an example one can consider the wide sense stationary and homogeneous isotropic noise presented in the beginning of this section. For this special situation, (27) takes the form

$$K_{o_{mn}}(\tau) = = K_o(\tau)\, (\lambda_o/2)\, \{\cos[(m-n)\pi] \int_{-\infty}^{+\infty} \mathrm{sinc}[(k-nk_L)L/2]\, \mathrm{sinc}[(k-mk_L)L/2]\, dk\,\} \,;$$

if the process $y_o(.)$ with correlation function $K_o(.)$ is narrow band in the vicinity of w_o and if the aperture lenght L is much larger than the carrier wavelenght λ_o, one can write

$$K_{o_{mn}}(\tau) \tilde{=} [\lambda_o/(2L)]\, K_o(\tau)\, \delta_{mn}$$

and $M=L/\lambda_o$.

Clearly, the covariance matrix of the vector process defined as

$$y_M(t) = [\, y_{o,-M}(t) | \ldots | y_{o,o}(t) | \ldots | y_{o,M}(t)\,]^T$$

with elements $K_{o_{mn}}(.,.)$ is positive definite. So, the matrix whose elements are the factor appearing between brackets {.} in (27) is factorable and $y_M(t)$ can be written as a function of $y_o(t)$

$$y_M(t) = H_o(t) \quad y_o(t) .$$

Once the dimension M of the truncated series (24) is fixed (M shall be chosen such that the MSE is "small enough"), one can form the (1x2M+1) vector

$$H_M(l)=\left[e^{jMk_L l} \mid \dots \mid 1 \mid \dots \mid e^{-jMk_L l} \right] , \quad l\varepsilon\Omega$$

and the random field y(t,l) is approximately given by

$$y(t,l) \overset{\sim}{=} y_M(t,l) = H_y(t,l) \; y \; (t) \; , \quad t\geq T_i \; , \; l \quad ,$$

where $H_y(t,l)$ is defined as

$$H_y(t,l) = H_M(l) \; H_o(t) \; H'(t)$$

and y(t) is governed by the ODE (23-a).

IV. FILTERING ALGORITHM IN THE PRESENCE OF NON WHITE OBSERVATION NOISE

One takes the observation field given by (10) and adds to it a non white observation noise; so, over the line aperture Ω , the observation field is

$$z(t,l) = H(t,l) \; x(t,l) \; +$$
$$+ \; H_y(t,l) \; y \; (t) \; + w(t,l) \; , \; t \geq T_i, \; l\varepsilon\Omega \; , \qquad (29)$$

where the vector propagating field x(.,.) is governed by equations (14) and (16), and y(t) is the solution of the ODE (23-a). It is assumed that y(t) is statisticaly independent of x(.,.) and w(.,.) and all the hypotesis in section II are maintained. The generalization of the filtering algorithm presented in section II when the observations are given by (29) is straightforward, although involving complicated algebraic manipulations. The filter equations are the following :

$$\left[\frac{\partial}{\partial t} + T(l|\underline{r}) \frac{\partial}{\partial l} \right] \hat{x}(t,l|Z_{t,\Omega}) =$$
$$= \int_\Omega P_{xx}(t;l,\sigma) \; H^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \; +$$
$$+ \; P_{xy}(t,l) \int_\Omega H_y^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \; , \qquad (30\text{-a})$$

$$\frac{d\hat{x}_o(t|Z_{t,\Omega})}{dt} = F(t) \; \hat{x}_o(t|Z_{t,\Omega}) \; +$$
$$+ \int_\Omega P_{xx}(t;l_o,\sigma) \; H^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \; +$$
$$+ \; P_{xy}(t;l_0) \int_\Omega H_y^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \quad , \qquad (30\text{-b})$$

$$\frac{d\hat{y}(t|Z_{t,\Omega})}{dt} = F'(t) \; \hat{y} \; (t|Z_{t,\Omega}) \; +$$
$$+ \int_\Omega P_{yx}(t,\sigma) \; H_y^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \; +$$
$$+ \; P_{yy}(t) \int_\Omega H_y^T(t,\sigma) \; R^{-1}(t,\sigma) \; V(t,\sigma) \; d\sigma \quad , \qquad (30\text{-c})$$

where the error covariance matrices are defined as

$$P_{xx}(t;l,\lambda) = E\{ \; \tilde{x}(t,l|Z_{t,\Omega}) \; \tilde{x}^T(t,\lambda|Z_{t,\Omega}) \; \} \; ,$$

$$P_{xy}(t,l) = E\{ \; \tilde{x}(t,l|Z_{t,\Omega}) \; \tilde{y}^T(t|Z_{t,\Omega}) \; \} = P_{yx}^T(t,l) \; ,$$

$$P_{yy}(t) = E\{ \; \tilde{y} \; (t|Z_{t,\Omega}) \; \tilde{y}^T(t|Z_{t,\Omega}) \; \} \; ,$$

and the innovations field is now

$$V(t,l) = z(t,l) - H(t,l)\hat{x}(t,l|Z_{t,\Omega}) -$$
$$- \; H_y(t,l)\hat{y}(t|Z_{t,\Omega}) \quad .$$

As in section II, all the error covariance matrices can be computed offline by integrating some Riccati PDE's of the type of (21). Due to space limitations one does not write them here but rather emphasizes that they can be easily obtained by writing the differential equations of the estimation errors and then proceeding on applying the definitions above.

V. CONCLUSIONS

In the present paper one presented a time recursive filtering algorithm for propagating random fields that verify the homogeneous wave equation when they are observed over a line aperture within additive white and non directional noise fields. The main problem here was that of suitably modelling the non directional noise component. The spatial Fourier expansion approach was directed to get a simple time variant state space model with an observation matrix depending of the harmonically related complex exponential functions $\exp(j2\pi \, ml/L)$. The cost to pay for simplicity is the possible large dimension of the state vector y(.). However, one claims the generality of this model which, under the stationary and narrow band assumptions, conveniently aproximates those presented by Baggeroer(c.f.).

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