WAVENUMBER ESTIMATION STATISTICS FOR A GENERALIZED ESTIMATOR

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RESUME

Dans bon nombre d'applications, des groupes d'éléments sont utilisés pour déterminer les propriétés des ondes en propagation. Dans le radar et le sonar, par exemple, un groupe d'éléments est utilisé pour déterminer le contenu du spectre et les coordonnées spatiales des cibles ainsi que le bruit dans le champ en observation. Dans bien des cas, les évaluations des paramètres spectraux et spatiaux sont faites d'après une évaluation approximative du spectre de la matrice pour la sortie du groupe. Le paramètre spatial présentant de l'intérêt est habituellement le numéro d'onde vecteur de l'onde de propagation.

Nous considérons ici une évaluation généralisée du numéro d'onde et nous établissons les propriétés statistiques asymptotiques de l'évaluation. En utilisant les résultats bien connus de normes asymptotiques pour les valeurs eigen et les vecteurs eigen approximativement évalués des matrices de spectre, nous prouvons que l'évaluation approximative de la formule de l'onde est asymptotiquement normale et nous calculons l'écart avec la distribution à la limite.

SUMMARY

In many applications arrays of sensors are used to determine the properties of propagating waves. In radar and sonar, for example, an array of sensors is used to determine the spectral content and spatial coordinates of targets and noise in the observed field. In many situations estimates of the spectral and spatial parameters are based on the estimated cross-spectral matrix for the array output. The spatial parameter of interest is usually the vector wavenumber of the propagating wave.

In this paper we consider a generalized wavenumber estimator and establish asymptotic statistical properties for the estimator. Using the well known asymptotic normality results for the estimated eigen-values and eigenvectors of cross-spectral matrices we prove that the wavenumber estimate is asymptotically normal and calculate the variance of the limiting distribution.

I. INTRODUCTION

Consider an arbitrary three-dimensional array of point receiving sensors. The array is in a medium with a three-dimensional noise field and, using spectral estimation methods, is used to estimate the vector velocity, or wavenumber, of propagating waves in the medium.

It is well known that a stationary random process can be characterized by means of a spectral density function. This function provides information concerning the power as a function of frequency. In a similar manner, propagating waves in a homogeneous random field can be characterized by a frequency-wavenumber spectral density function. This function provides information concerning the power as a function of frequency and the vector velocity of the propagating waves. In this paper we consider the situation where the frequency is assumed to be known and consider estimation of the wavenumber spectra only.

We assume that the array consists of M omni-directional sensors; we let $\Phi(\omega)$ be the M×M cross-spectral matrix for the array at frequency ω. Its estimate will be denoted by $\hat{\Phi}^N(\omega)$. A variety of frequency-wavenumber estimators, some based on the estimated cross-spectral matrix, has been proposed, see [1], [2], and [3]. In this paper we consider a generalized form of the frequency-wavenumber estimator, and develop asymptotic statistics for this estimator. Our estimate is similar to that proposed by Pisarenko [3] except for the fact that our estimate is based on the more standard smoothed cross-spectral matrix estimate and we eliminate the Gaussian assumption on the process imposed in [3].

In Section II we present the standard estimate of the smoothed cross-spectral matrix. We also present the form of the generalized frequency-wavenumber estimate. In Section III we present Brillinger's [4] results on asymptotic statistics for eigenvector and eigenvalue estimates for smoothed cross-spectral density matrix estimates. We also present our results on asymptotic statistics for a generalized wavenumber estimate. In Section IV we comment on the dependence of the asymptotic variance on array configuration and the eigenvalues of the cross-spectral matrix.

II. THE GENERALIZED WAVENUMBER ESTIMATE

We assume that the output of a sensor, say the j^{th} located at the vector position $\underline{x}_j$, is a wide sense stationary discrete-time random process with zero mean. We denote the output process for the j^{th} sensor by $\{z_{j,n}\}_{n=-\infty}^{\infty}$.

Given a finite set of observations for the j^{th} sensor output process, $\{z_{j,n}\}_{n=1}^{N}$, we form the discrete Fourier transform

$$\hat{Y}_j^N(2\pi m/N) = \sum_{n=0}^{N-1} z_{j,n}\exp(-i2\pi nm/N) \qquad m = 1,2,\cdots,N$$

and for a pair of sensors, say j and k, we calculate the second order periodogram

$$\hat{I}_{j,k}^N(2\pi m/N) = (1/2\pi N)\hat{Y}_j^N(2\pi m/N)\hat{Y}_k^N(2\pi m/N)^*$$

and the smoothed cross-spectral density estimate

$$\hat{\phi}^N_{j,k}(\omega) = (2\pi/N)\sum_{m=1}^{N} a^N(\omega - 2\pi m/N)\hat{I}^N_{j,k}(2\pi m/N)$$

where $a(\alpha)$, $-\infty < \alpha < \infty$, is a weight function such that

$$a^N(\alpha) = \sum_{m=-\infty}^{\infty} a[M_N(\alpha + 2\pi m)]$$

with $\int_{-\infty}^{\infty} a(\alpha)d\alpha = 1$ and $\int|a(\alpha)|d\alpha < \infty$. M_N (bandwidth parameter) is a sequence of positive numbers such that $M_N \to \infty$ and $M_N/N \to 0$ as $N \to \infty$. The cross-spectral density matrix estimate is given by

$$\hat{\Phi}^N(\omega) = \left[\hat{\phi}^N_{j,k}(\omega)\right]_{\substack{j=1,2,\cdots,M\\ k=1,2,\cdots,M}} . \qquad (1)$$

We note that many authors in order to simplify the analysis do not take the weight function, $a(\alpha)$, into account in their estimate. Even though it complicates the evaluation of asymptotic statistics it must be included in order to achieve a consistent estimate of the cross-spectral density matrix.

Since $\Phi(\omega)$ is a Hermitian matrix, the usual eigenvalue and eigenvector decomposition applies and we can write

$$\Phi(\omega) = \sum_{j=1}^{M} \mu_j(\omega)\underline{U}_j(\omega)\underline{U}_j^*(\omega) \qquad (2a)$$

and

$$\hat{\Phi}^N(\omega) = \sum_{j=1}^{M} \hat{\nu}_j(\omega)\hat{\underline{V}}_j(\omega)\hat{\underline{V}}_j^*(\omega) \qquad (2b)$$

where $\{\mu_j(\omega)\}_{j=1}^M$ and $\{\underline{U}_j(\omega)\}_{j=1}^M$ are the eigenvalues and eigenvectors of $\Phi(\omega)$. The sets $\{\hat{\nu}_j(\omega)\}_{j=1}^M$ and $\{\hat{\underline{V}}_j(\omega)\}_{j=1}^M$ are the estimated quantities.

Now let $g(z)$ be an analytic function on the half-plane $Re(z) > 0$ and monotonically increasing function of x for $x > 0$, define $G(x) = g^{-1}(x)$, then $G(z)$ is also analytic for $Re(z) > 0$ and monotonically increasing function of x for $x > 0$.

We now define the generalized wavenumber estimate at frequency ω as $\hat{P}^N(\omega,\underline{\kappa})$ by

$$\hat{P}^N(\omega,\underline{\kappa}) = G\{\underline{D}_{\underline{\kappa}}^* g[\hat{\Phi}^N(\omega)]\underline{D}_{\underline{\kappa}}\} \qquad (3)$$

where the "steering" vector $\underline{D}_{\underline{\kappa}}^*$ is given by

$$\underline{D}_{\underline{\kappa}}^* = [w_1\exp(-i\underline{\kappa}\cdot\underline{x}_1),\ w_2\exp(-i\underline{\kappa}\cdot\underline{x}_2),\ \cdots,\ w_M\exp(-i\underline{\kappa}\cdot\underline{x}_M)] \qquad (4)$$

and the set $\{w_j\}_{j=1}^M$ is wavenumber window weights, the set $\{\underline{x}_j\}_{j=1}^M$ is sensor position vectors, and $\underline{\kappa}$ is the wavenumber vector, $\underline{\kappa} = (\omega/c)\underline{r}$, $\underline{r}$ is a vector in the direction of a propagating wave and c is the velocity of propagation.

Using the eigen-decomposition of (2b) we can re-write the generalized wavenumber estimate $\hat{P}^N(\omega,\underline{\kappa})$ of (3) as

$$\hat{P}^N(\omega,\underline{\kappa}) = G\left\{\underline{D}_{\underline{\kappa}}^*\left[\sum_{j=1}^{M} g(\hat{\nu}_j(\omega))\hat{\underline{V}}_j(\omega)\hat{\underline{V}}_j^*(\omega)\right]\underline{D}_{\underline{\kappa}}\right\}. \qquad (5)$$

Letting $g(x) = x$ in (5) yields the conventional wavenumber estimate, that is

$$\hat{P}^N(\omega,\underline{\kappa}) = \underline{D}_{\underline{\kappa}}^*\left[\sum_{j=1}^{M} \hat{\nu}_j(\omega)\hat{\underline{V}}_j(\omega)\hat{\underline{V}}_j^*(\omega)\right]\underline{D}_{\underline{\kappa}} . \qquad (6)$$

If we let $g(x) = 1/x$ in (5), we obtain the minimum energy estimate of Capon [1], that is

$$\hat{P}^N(\omega,\underline{\kappa}) = \left\{\underline{D}_{\underline{\kappa}}^*\left[\sum_{j=1}^{M} (1/\hat{\nu}_j(\omega))\hat{\underline{V}}_j(\omega)\hat{\underline{V}}_j^*(\omega)\right]\underline{D}_{\underline{\kappa}}\right\}^{-1} . \qquad (7)$$

In the next section we examine asymptotic statistics for the generalized estimate of (5).

III. ASYMPTOTIC STATISTICS

In order to estimate the asymptotic statistics of the generalized wavenumber estimator of (5) we rely on the asymptotic statistics for eigenvalues and eigenvectors of smoothed cross-spectral matrices as developed by Brillinger [4].

We define the vector $\underline{\theta}(\omega)$ and its estimate $\hat{\underline{\theta}}^N(\omega)$ by

$$\underline{\theta}(\omega) = \left[\mu_1(\omega),\ \cdots,\ \mu_M(\omega), \underline{U}_1(\omega),\ \cdots,\ \underline{U}_M(\omega)\right] \qquad (8a)$$

$$\hat{\underline{\theta}}^N(\omega) = \left[\hat{\nu}_1(\omega),\ \cdots,\ \hat{\nu}_M(\omega), \hat{\underline{V}}_1(\omega),\ \cdots,\ \hat{\underline{V}}_M(\omega)\right] \qquad (8b)$$

The following assumption on the summability of all order cumulants of the processes $z_{j,n}$, $j = 1, 2, \cdots, M$ is a fundamental assumption of Brillinger [4] required for establishing asymptotic statistics for second order periodograms.

Assumption 1: All order cumulants of $z_{j,n}$, $j = 1, 2, \cdots, M$, satisfy

$$\sum_{r_1=-\infty}^{\infty}\cdots\sum_{r_{m-1}=-\infty}^{\infty} \{1 + |r_j|\}\left|c_{b_1,\cdots,b_m}(r_1,\cdots,r_{m-1})\right|<\infty$$

for $j = 1, 2, \cdots, m - 1$ and any m-tuple $b_1, \cdots, b_m$ when $m = 2, 3, \cdots$.

The next assumption is also a fundamental assumption. This one is required for establishing asymptotic statistics for eigenvalues and eigenvectors of cross-spectral matrices.

Assumption 2: All eigenvalues of the $M\times M$ array cross-spectral matrix are distinct.

The following theorem, proven by Brillinger [4], establishes asymptotic normality for the vector of eigenvalues and eigenvectors, $\underline{\hat{\theta}}^N(\omega)$.

Theorem 1: Let $\underline{z}_n$ be a stationary vector process of sensor outputs with components $z_{j,n}$, $j = 1, 2, \cdots, M$, satisfying Assumption 1. Furthermore, let $\Phi(\omega)$ be the cross-spectral matrix for the vector process $\underline{z}_n$ and let the eigenvalues of Φ satisfy Assumption 2. Then $\underline{\hat{\theta}}^N(\omega)$ of (8b) is asymptotically normal with asymptotic mean $\underline{\theta}(\omega)$ and asymptotic covariance structure given by

$$\lim_{N\to\infty} \operatorname{cov}\left\{(N/M_N)^{1/2}(\underline{\hat{\theta}}^N(\omega)-\underline{\theta}(\omega)),(N/M_N)^{1/2}(\underline{\hat{\theta}}^N(\omega)-\underline{\theta}(\omega))\right\}$$

$$= \sum$$

where $\sum$ is given by

$$2\pi \int_{-\infty}^{\infty} a(\alpha)^2 d\alpha \begin{bmatrix} \begin{matrix} \mu_1^2 & & 0 \\ & \ddots & \\ 0 & & \mu_M^2 \end{matrix} & & & 0 \\ & \mu_1 \sum\limits_{m\neq 1} \frac{\mu_m \mathbf{U}_m \mathbf{U}_m^*}{(\mu_1-\mu_m)^2} & & \\ & & \ddots & \\ 0 & & & \mu_M \sum\limits_{m\neq M} \frac{\mu_m \mathbf{U}_m \mathbf{U}_m^*}{(\mu_M-\mu_m)^2} \end{bmatrix}$$

for $\omega \neq n\pi$, $n = 1, 2, \cdots$, the structure of Σ is more complicated if $\omega = n\pi$, but is not presented.

We now state, without proof, the following theorem which shows that the generalized estimate of wavenumber, (5), is asymptotically normal and provides the structure of the asymptotic variance. The proof follows from the result of Theorem 1 and from the general result for functions of asymptotically normal random vectors, see Serfling [5]. Note that this result is a convergence in distribution result, and, thus, does not imply the existence of moments of $\hat{P}^N(\omega,\underline{\kappa})$.

Theorem 2: Under Assumptions 1 and 2, and under the assumption that $N/M_N^5 \to 0$ as $N \to \infty$, we have

$$(N/M_N)^{1/2}\, \hat{P}^N(\omega,\underline{\kappa}) - P(\omega,\underline{\kappa}) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\sigma^2(\omega,\underline{\kappa}))$$

where for $\omega \neq \bmod \pi$ the variance $\sigma^2(\omega,\underline{\kappa})$ is

$$\sigma^2(\omega,\underline{\kappa}) = 2\pi \int_{-\infty}^{\infty} a^2(\alpha)d\alpha \left[G'\left(\sum_{k=1}^{M} g(\mu_k(\omega))B_k^2(\omega,\underline{\kappa})\right)\right]^2$$

$$\left[\sum_{j=1}^{M} g'(\mu_j(\omega))^2\mu_j^2(\omega)B_j^4(\omega,\underline{\kappa}) + 8\sum_{j=1}^{M} g^2(\mu_j(\omega)) \sum_{m=j+1}^{M} \frac{\mu_j(\omega)\mu_m(\omega)B_j^2(\omega,\underline{\kappa})B_m^2(\omega,\underline{\kappa})}{(\mu_j(\omega)-\mu_m(\omega))^2}\right] \quad (10)$$

and $B_j^2(\omega,\underline{\kappa}) = |\mathbf{D}_{\underline{\kappa}}^* \mathbf{U}_j(\omega)|^2$.

IV. DISCUSSION

We see from the result of Theorem 2 that the variance of the asymptotic distribution for the generalized wavenumber estimator, (3), tends to zero as $N \to \infty$. We also see from the variance expression (10) that the asymptotic variance is dependent on the weight function $a(\alpha)$, the eigenvalues of the cross-spectral matrix $\Phi(\omega)$, the first derivatives of the functions G and g and the array response function $B_j(\omega,\underline{\kappa})$. It is apparent from (10) that the array response function $B_j(\omega,\underline{\kappa})$ is a major factor of the variance expression. We note that the contribution of the wavenumber weights $\{w_j\}_{j=1}^M$ is imbedded into the function $B_j(\omega,\underline{\kappa})$. The exact relationship between the asymptotic variance and the cross-spectral matrix eigenvalues is not clear at this time. The impact of the selection of an eigenvalue subset is also not clear at this time.

REFERENCES

[1] J. Capon, "High-Resolution frequency-wavenumber spectrum analysis," IEEE Proceedings, vol. 57, pp 1408-1418, 1969.

[2] D.H. Johnson and S.R. DeGraaf, "Improving the resolution of bearing in passive sonar arrays by eigenvalue analysis," IEEE Trans. Acoust., Speech, Signal Processing, vol ASSP-30, pp 638-647, 1982.

[3] V.F. Pisarenko, "On the estimation of spectra by means of non-linear functions of the covariance matrix," Geophys. J. R. Astr. Soc., vol 28, pp 511-531, 1972.

[4] D.R. Brillinger, Time Series: Data Analysis and Theory. New York, NY: Holt, Rinehart and Winston, Inc., 1975.

[5] R.J. Serfling, Approximation Theorems of Mathematical Statistics. New York, NY: Wiley and Son, Inc., 1980.