HIGH RESOLUTION WIGNER-VILLE ANALYSIS

Boualem BOASHASH* and Harper J. WHITEHOUSE**

*CRISSP, UNIVERSITY OF QUEENSLAND, BRISBANE, 4067, AUSTRALIA
**NAVAL OCEAN SYSTEMS CENTER, CODE 7402, SAN DIEGO, CA.92500/5000 USA

RESUME

Nous présentons une technique d'analyse Temps-Fréquence à haute résolution basée sur une version modifiée de la Représentation Temps-Fréquence de Wigner-Ville (WVD). Cela est obtenu en reconnaissant que la WVD est définie comme étant la Transformée de Fourier (TF) d'une forme bilineaire et complexe, puis en remplacant la TF par une méthode d'analyse spectrale paramétrique à haute résolution.

SUMMARY

We present a technique for high resolution time-frequency signal analysis based on a modified version of the Wigner-Ville Distribution (WVD). This is achieved by recognizing that the WVD is the Fourier Transform of a bilinear complex kernel and replacing the Fourier Transform by a high resolution model based spectral estimator.

1. INTRODUCTION:

Time-frequency signal analysis has recently become the focus of research as more and more analysts are confronted with the widely reported insufficiencies of classical signal analysis tools, based either on time-domain or frequency domain representations of the signal. Although Wigner's formulation dates back to 1932 [1] and Ville proposed its use in signal analysis in 1948 [2] at which time he defined the analytic signal, it was not until the late 1970's that research teams pioneered its actual use in engineering applications [3a] [3b]. These efforts in Europe benefitted from a general formulation of joint Time-Frequency Distributions (TFDs) proposed by L. Cohen [4], which provided a sufficiently comprehensive framework for the study and understanding of previously narrowly defined representations such as Rihaczek's, [5], Page's [6], the spectrogram [7], the sonogram [8], as well as those already mentioned which are today referred to as the Wigner-Ville Distribution (WVD).

A criterion for choosing the most appropriate TFD for engineering applications was proposed in 1978-1979 by Bouachache et al [3a], [3b], and two years later in 1980 by Claasen and Meklenbrauker [9]. This criterion consists of selecting the TFD which provides an unbiased estimate of the instantaneous frequency of the signal with the best resolution and accuracy in the Time-Frequency domain when applied to a monocomponent frequency modulation (FM) signal. Based on this criterion, Claasen and Meklenbrauker demonstrated in a series of papers in the Phillips Technical Journal that for real valued signals the Wigner Distribution was an "optimum" tool for time-frequency signal analysis despite the presence of low frequency artifacts. Bouachache (now spelled Boashash) argued that the analytic signal associated with the real signal under analysis should be used and therefore named this TFD the Wigner-Ville Distribution (WVD) in recognition of the contribution of Ville who first proposed the method for signal analysis [10a] [10b]. Since then, fast developments in this field have been observed regarding both theory and applications of the WVD as interest in the method spread among scientists. Successful applications to various areas have been reported, among them seismic [10], oceanography [11a] [11b], loudspeakers [12a] [12b]. Many other papers speculated about the possible applications of the method to a variety of situations, without going much further [13] [14]. Indeed, the number of successful applications is still very limited. There are several reasons for this:
1) The apparent complexity of the method, as reported in some papers which incorrectly

describe an "aliasing problem" which results from the use of the real instead of the analytic signal. [9b].
2) Confusion concerning the criterion upon which the choice of the WVD is based.
3) A problem with the windowing of the signal, when it is not of finite duration, or when it is desired to have a frequency resolution independent of the choice of a window.

We believe that the first two points can be overcome by careful application of the Wigner-Ville analysis procedure [15] [16]. As far as we know, point number 3 has not been dealt with in much detail. Therefore, the primary object of this paper is to discuss this latter issue and to propose a new method for Time-Frequency signal analysis using a modified Wigner-Ville Distribution [15][17].

2. THE WINDOWED WIGNER-VILLE DISTRIBUTION:

Let $x(t)$ be a real function of continuous-time with Fourier Transform $X(f)$. The analytic signal associated with $x(t)$ may be expressed [2] as:

$$z(t) = x(t) + j H [x(t)] \qquad (1)$$

where H denotes the Hilbert Transform defined by

$$H [x(t)] = p.v. \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\zeta)}{t - \zeta} d\zeta \qquad (2)$$

The Fourier Transform, $Z(f)$, of $z(t)$ is related to $X(f)$ as follows:

$$Z(f) = \begin{cases} 2X(f), & f > 0 \\ X(f), & f = 0 \\ 0, & f < 0 \end{cases} \qquad (3a)$$

where $X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} dt \qquad (3b)$

The Wigner-Ville Distribution (WVD) is defined [2] as:

$$W_z(t,f) = \int_{-\infty}^{+\infty} z(t + \tau/2)z^*(t - \tau/2)e^{-j2\pi f\tau} d\tau \qquad (4)$$

Equation (4) implies that evaluation of the WVD is a non-causal operation, in the same sense as Eqs. (2) and (3b) imply that these transforms are non-causal. Thus, the value of the signal must be known for all time before either the Hilbert transform or the WVD can be evaluated. This is a problem similar to the one observed for the Fourier transform (FT) and may be overcome by well-known techniques. The Hilbert transform can be approximated by the use of either

finite impulse response (FIR) filters or infinite impulse response (IIR) filters. In all of these cases, the limitation is overcome by applying the WVD to a windowed version of the signal. If the WVD is required at time t_a, the windowed signal is:

$$z_m(t,t_a) = z(t) \, m(t-t_a) \qquad (5)$$

where $m(t)$ is the window function and satisfies: $m(t) = 0$ for $t > |T_m/2|$ where T_m is the width of the analysis window. The WVD of the windowed signal is given [18] by a one-dimensional convolution in the frequency variable:

$$W_{z_m}(t,f) = [W_z * W_m](t,f) \qquad (6)$$

Thus the effect of windowing is to smear the WVD representation in the frequency direction only, hence the frequency resolution can be increased by using a longer window without adversely affecting the time resolution.

By evaluating $W_z(t,f)$ at time t_a, a "cross section" of the WVD at time t_a can be approximated. With a change of variable from t_a to t and using the shift invariance property of the WVD [3], the WVD at any point in time can be evaluated by shifting the signal $z(t)$ so that time t is mapped to the time origin. Therefore, the "cross section" of the WVD is evaluated about a point in time using:

$$W_{z_m}(f) = \int_{-\infty}^{+\infty} k(\tau)w(\tau)e^{-j2\pi f\tau}d\tau \qquad (7)$$

where $\quad k(\tau) = z(\tau/2)z^*(-\tau/2) \qquad (8)$

and $\quad w(\tau) = m(\tau/2)m^*(-\tau/2) \qquad (9)$

From Eq. (7) it can be seen that the WVD is effectively the FT of a kernel signal $k(\tau)$ which has been windowed by $w(\tau)$. Even though the time-domain window function is $m(\tau)$, the effective window in the bilinear kernel domain is $w(\tau)$ and hence the resulting frequency resolution is directly related to the spectrum of $m(\tau)$[18]. When a window is specified for the WVD it will be implied that this is the function $w(\tau)$.

As outlined in the literature [3][9][10], Wigner-Ville analysis provides a robust means of estimating the instantaneous frequency of a non-stationary deterministic signal such as the linear FM signal

$$s(t) = a(t).\cos\phi(t)$$

where $\quad \phi(t) = 2\pi(f_o t + \alpha t^2)$

Wigner-Ville analysis (WVA) may be thought of as a two-step process:

1) A sequence of kernels $k(t,\tau) = z(t+\tau/2).z^*(t-\tau/2)$ regarded as function of τ parameterized by t, is formed using the analytic signal $z(t)$ associated with the real signal $s(t)$. Since $s(t)$ obeys Bedrosian's conditions [3] then $H[s(t)] = a(t).\sin\phi(t)$, then the analytic signal becomes $z(t) = a(t).e^{j\phi(t)}$.

2) A Fourier transform of the bilinear kernel is taken with respect to the variable τ, using a window, $w(\tau)$, to reduce spectral leakage. For the linear FM signal, the bilinear kernel

$$k(\tau) = z(t + \tau/2).z^*(t - \tau/2)$$

becomes a complex sinusoid with frequency proportional to the chirp parameter α. A coarse estimate of the instantaneous frequency is provided by the position of the frequency peak in the magnitude of the kernel's Fourier Transform. However, for short integration times, i.e. windows, the FT is known to be a poor spectral estimator

[21]. In addition the window function further reduces the resolution of the Fourier Transform. In this paper, therefore, we propose to substitute a high resolution, model-based spectral estimator such as Pisarenko spectral analysis [19a] or the eigenvector-based spectral estimators such as those of Bienvenu and Kopp [19b], the MUSIC method of Schmidt [20a], or the ESPRIT technique of Paulraj, Roy and Kailath [20b]. If in some applications, these estimators are found to be computationally too intensive since they are of $O(n^3)$, then a carefully chosen version of the maximum entropy method originally proposed by Burg [21] could be used since it is of $O(n^2)$. In this case, care must be exercised to avoid potential problems with line splitting and bias errors [22].

3. THE MODIFIED WVD (MWVD):

A variety of numerical methods for super-resolution spectral estimation have been proposed [24b]. Since a harmonic process can be generated by an autoregressive (AR) model, the kernel $k(\tau)$ at time t may be fit by the output of an appropriate AR model; the AR coefficients will estimate the cross-section of the Wigner-Ville Distribution at time t. This method is based on extrapolation of a segment of a known autocorrelation function for the unknown lags. The extrapolated autocorrelation function satisfying the maximum entropy criterion will be the most random one consistent with the known segment of the autocorrelation lags corresponding to the kernel. Equivalently, the estimated spectrum is the smoothest one of all the spectra which do not conflict with the known segment of the autocorrelation lags. The direct application of this method is subject to several well known difficulties [23] which are:
- The choice of the order of the AR model;
- Noise influence; Since the method is based on a least-squares fitting of the kernel to the AR model, it is sensitive to noise which is additive with the signal.

Following Tufts and Kumaseran [22] and Minami et al [23], we modify the method to improve its robustness to the noise and to make it insensitive to the order of the AR model. This is achieved by using the singular value decomposition (SVD) of the kernel data matrix to compute a reduced rank pseudo-inverse for use in the AR model fitting process. This method of super-resolution described in [23] assumes the sampled kernel data under analysis can be expressed as a Pth order AR model [24a]:

$$C_t(k)= - \sum_{i=1}^{p} a_p(i) \, C_t(k-i) + n(k); \; k=p, \; N-1 \qquad (10)$$

where $C_t(k) = z(t+\lambda) . z^*(t-\lambda)$, $\lambda = k-(\frac{N-1}{2})$, N odd (11)

N is the number of input samples; $a_p(i)$, $i = 1 \, p$ are the coefficients of the AR model, and $n(k)$ is a gaussian white noise process.

(Here, we have assumed a change of variable $k = \tau/2$ before discretizing, as reported in [3].)

As previously mentioned, the spectral analysis of the sampled kernel data could be achieved using one of the various methods listed in [24b], such as Maximum Entropy Method, Prony's method, etc. All these methods are based on a solution of (13) which gives the best least-square fit while allowing for edge effects. To be efficient, an estimator must also be stable and converge quickly to the solution. There are several methods which provide the desired performance. The singular value decomposition (SVD) was adopted in this paper since it was shown to be insensitive to the model order and to improve noise immunity [23].

(It is important here to note a significant error in

[23]. The equations (1) and (9) of that paper are incompatible. If the equation (9) is to be used then the left part of equation (1) should be multiplied by (-1).). At time t, Equation (10) can be expressed as:

$$-[C].[a] + [n] = [c] \qquad (12)$$

with the following notations.

$$[a] = \begin{bmatrix} ap(p) \\ \vdots \\ ap(1) \end{bmatrix} \quad [n] = \begin{bmatrix} n(p) \\ \vdots \\ n(N-1) \end{bmatrix} \quad [c] = \begin{bmatrix} c(p) \\ \vdots \\ c(N-1) \end{bmatrix} \qquad (13)$$

$$[C] = \begin{bmatrix} c(o) & c(1) & \cdots & c(p-1) \\ c(1) & c(2) & & c(p) \\ c(2) & & & \bullet \\ \vdots & \bullet & & \bullet \\ c(N-P-1) & & & c(N-2) \end{bmatrix}$$

In the absence of noise, the estimate $[\hat{a}]$ (ie, AR coefficients) can be given by:

$$[\hat{a}] = -[C]^+ . [c] \qquad (14)$$

where $[C]^+$ is the generalized inverse of the matrix [C] obtained via the reduced rank SVD.

This solution corresponds to a least-square fit of the data to the model chosen.

The cross-section of the Wigner-Ville Distribution at frequency f is then estimated by:

$$W(t,f) = \frac{\sigma_n^2}{\left|1 + \sum_{i=1}^{p} \hat{a}_p(i)\exp(-j2\pi fi)\right|^2} \quad , o< f <1/2 \qquad (15)$$

where σ^2 represents the variance of n(k).

The pseudoinverse is used instead of $[C]^{-1}$ since in many practical cases, the matrix [C] is singular and it's inverse does not exist.

The modified Wigner-Ville distribution estimated by (15) should improve further the spectral resolution achievable by the results presented in the next section.

4. IMPLEMENTATION OF THE ALGORITHM:

4.1. The Proposed Algorithm

Currently Wigner-Ville analysis is performed using a software package developed by B. Boashash, L. White and colleagues for use in the CRISSP laboratory [25]. The software package can analyze signals from user supplied files or from the built-in signalgenerator. Results of signal analysis are then processed by a graphics package to provide the user with a Time-Frequency plot. In the WVD package, a sliding window is applied to the sampled data, and the Wigner-Ville analysis performed on each windowed portion. For each windowed portion, the windowed kernel $k(\tau).w(\tau)$ is formed and then passed to an FFT algorithm to calculate the "spectrum". A Flowchart for the WVD algorithm is shown in [25].

In the new method, the singular value decomposition is used to form the generalized inverse $[C]^+$ of the observation kernel matrix [C]. The estimate of the AR coefficients are formed from $[C]^+ . [c]$, and the Power Spectral Density (PSD) from (15). A facility to calculate the generalized inverse is available in the mathematics package MATLAB [26], so it was decided to link these routines into the WVD source code. The original MATLAB package is configured to deal with a total of approximately 5,000 matrix

elements (i.e., 2,500 complex elements). This limited the size of the observation matrix [C] to approximately 20 x 40 complex samples. The MATLAB source code has then been subsequently altered [25] to allow for up to 32,000 real elements which now limits the SVD to matrices up to 40 x 100 complex samples. The algorithm to calculate the Power Spectral Density (PSD) given N input samples is entirely implemented by MATLAB. The sequence of steps to be followed is shown in the theory section and the equivalent steps in MATLAB are shown in [25]. Unless otherwise stated, the order of the model P is N/3 where N is the number of samples input to the PSD estimator. The value of N is not usually constant for each windowed portion as it depends on the number of samples available either side of the point in time of interest. If there are M samples available either side of "itime", then a window length 2M is used. Otherwise only the length corresponding to the available samples are used.

5. RESULTS:

5.1 Initial testing and integration into TFD.

To verify the AR modelling approach presented in [23], the SVD was initially applied to find the PSD of a 64 point sinewave of frequency , where f_s is the sampling frequency. The SVD was also applied to a 64 point sinewave at $f_s/8$ using 48, 50, 100, and 200 frequency beams between 0 and $f_s/2$. The results show that resolution increases as more points are used, but if the input frequency lies between two frequency beams, then it will be missed [25].

The next step investigated the performance of AR modelling in place of the FFT when using Wigner-Ville analysis by substituting the SVD and FFT calculations into the TFD package currently used by CRISSP. (The integration into TFD was performed in 3 stages to minimize errors and enable an independent check of intermediate results so as to minimize the risk of multiple errors cancelling each other and to efficiently localize the cause of these errors [25]). Tests were first performed on real windowed signals. The SVD spectral estimator performed very well in localizing the spectral components of the signal, but has trouble coping with a small model order when windowing is applied [25]. Then further testing was done using the windowed analytic signal. As expected, the SVD appears to have had no problems in resolving the spectrum except at small window lengths when using a Hanning window [25].

5.2 Testing the windowed kernel.

The MWVA was initially applied to 64 samples of a single sinewave of frequency 25 Hz, sampled at 200 Hz. The MWVA resolved the frequency of the signal with an extreme accuracy and markedly improved the one obtained by the WVD [25].
The superior performance of the Wigner-Ville Distribution as a tool for Time-Frequency Signal Analysis was first demonstrated on linear FM signals [3]. The MWVA should, therefore, perform even better than the WVD for this class of signals. The test signal is a 64 sample linear FM signal sampled at 200 Hz and modulated from a lower frequency of 10 Hz to an upper frequency of 80Hz. The results are shown in Figs 1 and 2.

The improvement in spectral resolution achievable using the MWVA over conventional WVD can be seen by comparing these results with those obtained in Figs. 1 and 2. The improvement in spectral resolution at short window lengths is quite marked. The major problems with this approach are the spurious peaks observed in the plots as explained in the previous section. The MWVA of the linear FM signal was calculated with various window sizes with the PSD estimator determined for 200, 400, 600 and 800

spectral lines using normalized plots. Only 7 traces per plot were obtained. This was considered sufficient to demonstrate the validity of the theory presented as the FM law is determined remarkably easily and with an extremely good accuracy.

It was observed that increasing the spectral resolution from 200 to 800 lines did not eliminate all the spurious peaks.

6. DISCUSSION AND CONCLUSION:

This paper has demonstrated how a modified Wigner-Ville Analysis could substantially improve the spectral resolution achieved by the WVD. The high-resolution spectrum estimator incorporated is arbitrary, hence the problems of spurious peaks were ignored for the time being. Once we have established the plausability argument, it is then just a matter of optimizing the algorithm for estimating the PSD. This can be done in several ways, either by choosing another method such as Prony's method or by improving the SVD based PSD estimator. The argumentation for choosing Prony's method is that the SVD spectral estimator is not linear, which means that the relative amplitudes of spectral peaks are not preserved, when one varies the size of the window. In most applications, it is important that this property be observed. Nonetheless, the initial results obtained were quite encouraging and further investigation of the existing MWVA algorithm is warranted so as to optimize this method of Time-Frequency Signal Analysis by:
a) Varying the order of the model P, and checking the limit of the insensitivity of the model, that is define a precise rule for an optimum choice of P.
b) Varying the threshold below which singular values are discarded.

In the results presented, the model order was set to a value of N/3, where N is the number of samples in the windowed portion of the signals.
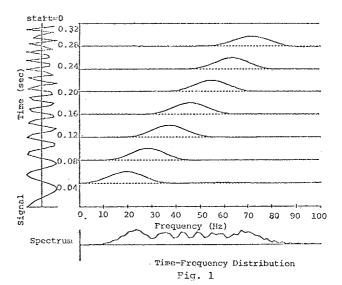
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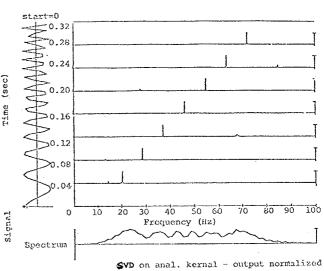[26] PC-MATLAB, 2.0, 1985, Math Works Inc.

Time-Frequency Distribution
Fig. 1

SVD on anal. kernal - output normalized
Fig. 2