THE COVARIANCE DIFFERENCE METHOD IN SIGNAL DETECTION

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RESUME

La méthode de différence de matrices de covariance est une addition récente aux méthods de détection basées sur l'espace des signaux. Elle offre l'avantage de nes pas présupposer un bruit blanc ou de covariance connue. On démontre que la matrice différence a autant de valeurs propres positives que négatives. Ce résultat suggère plusieurs modfications d'un test classique du maximum de vraisemblance pour déterminer la dimension de l'espace des signaux. Une méthode rectificative utilisant uniquement le niveau du bruit permet d'améliorer la sensibilité de ces tests.

ABSTRACT

The covariance difference method is a recent addition to the various signal subspace methods for signal detection that eliminates the requirement of a white or known noise covariance matrix. We show that the number of positive eigenvalues is the same as the number of negative eigenvalues, and this suggests several simple modifications of a well-known likelihood-ratio test for determining the dimension of the signal space. The sensitivity of these tests can be increased further by a rectification procedure utilizing only the noise level.

1. INTRODUCTION

The covariance-difference method [1,2] is an array-processing technique that permits use of the recently popularized signal subspace methods, such as MUSIC [3], without the need for the assumption that the noise is white or that it has a known covariance matrix. Instead, the method requires only that the noise be isotropic in the sense that if the array is rotated or shifted between two sets of measurements the noise covariance matrix should be the same in the two measurements. The requirement for isotropy of the background noise is much less stringent than the requirement that the noise covariance structure should be known, and it is a requirement that is often satisfied in practice. To use the method, the signals obtained from the array elements are stored for a predetermined observation period and are then used to form an estimate of the received-signal covariance matrix. The array is then rotated or otherwise moved, another set of measurements is stored and is used to form a second covariance-matrix estimate. The covariance matrices obtained from the two measurements are then subtracted to form the covariance-matrix difference. The general form of the difference is:

$$K_1 - K_2 = Q_1 - Q_2 + A_1 S_1 A_1^* - A_2 S_2 A_2^*$$

$$= Q_1 - Q_2 + (A_1, A_2) \begin{bmatrix} S_1 & \\ & -S_2 \end{bmatrix} (A_1, A_2)^* \qquad (1)$$

where Q_1 and Q_2 are the estimated noise covariance matrices obtained from the two measurements, A_1 and A_2 are the steering-vector matrices, S_1 and S_2 are the estimated signal covariance matrices, and the superscript * refers to Hermite transposition. Because the noise is assumed to be isotropic, the matrices Q_1 and Q_2 are the estimates of the same noise covariance matrix. Hence the true eigenvalues of $Q_1 - Q_2$ are zero. Also for p signals the rank of $A_1 S_1 A_1^* - A_2 S_2 A_2^*$ is no larger than 2p. It is less than 2p if any of the steering vectors in A_1 and A_2 are identical or linearly dependent. This can happen if the angle of rotation between the two sets of measurements coincides with the angular displacement between the sources.

2. PROPERTIES OF THE EIGENSPECTRUM

Theorem 1

If the matrix $A_1 S A_1^* - A_2 S A_2^*$ is full rank it has p positive and p negative eigenvalues.

To prove this let

$$C = [A_1, A_2, W]. \qquad (2)$$

where W is chosen so that C is a square, nonsingular matrix. This can always be done if $[A_1, A_2]$ has full rank. Also consider the diagonal matrix

$$D = \begin{bmatrix} S & & \\ & -S & \\ & & 0 \end{bmatrix} \qquad (3)$$

where 0 is a zero matrix of dimension M-2p. Then

$$CDC^* = A_1 S A_1^* - A_2 S A_2^* \qquad (4)$$

This is a congruence transformation, and by Sylvester's law of inertia [5] the matrix CDC^* has the same number of positive eigenvalues as D, the same number of negative eigenvalues, and the same number of zero eigenvalues. Thus if there are no steering-vector cancellations the eigenvalue structure of $K_1 - K_2$ consists of p positive eigenvalues, M-2p zero eigenvalues, and p negative eigenvalues. It is easy to give counterexamples to show that the theorem is not true if there are cancellations. The multiplicity of the zero eigenvalues can be used as an estimate of p. Because of the possibility of cancellations this estimate

is only a lower bound on the number of signal sources. The probability of cancellations can be reduced by rotating the array through several different angles.

3. A LIKELIHOOD-RATIO TEST

The statistical properties of eigenvalues and eigenvectors of sample covariance matrices have been examined in the literature [6,7], but results are generally restricted to eigenvalues of positive-definite covariance matrices. The likelihood-ratio test for the hypothesis H_0 that the eigenvalues λ_c, λ_{c+1}, ... λ_{c+d} are equal is

$$L = -nd \log (g/a) \qquad (5)$$

where n is the number of independent observations (snapshots),

$$g = \left[\prod_{i=c}^{c+d} \lambda_i\right]^{1/d}$$ is the geometric mean

and

$$a = (1/b) \sum_{i=c}^{c+d} \lambda_i$$ is the arithmetic mean of the eigenvalue estimates. When H_0 is true L is approximately χ^2_ν with $\nu = 1/2\ (d-1)(d+2)$.

Since we are dealing with a nonpositive-definite matrix, and since the eigenvalues of interest are, in fact, zero, this test is clearly not applicable. However the fact that for full rank $[\mathbf{A}_1, \mathbf{A}_2]$ the number of positive and negative eigenvalues is the same suggests applying the test to only the positive half of the eigenspectrum. Alternatively one can apply the test to the absolute values of the negative half of the eigenspectrum. Since the eigenspectrum is generally not symmetric the positive half might yield better sensitivity than the negative half, or vice versa. Thus the further possibility suggests itself of using the absolute value of all of the eigenvalues in the test to average the risk, so to speak. Prasad et al. [8] have recently demonstrated exact symmetry of the eigenspectrum for linear arrays with uniform element spacing and where the angle of rotation between measurements is 180^o. In this case there is clearly no need to consider anything beyond the positive half of the eigenspectrum.

The test is recursive, starting with the hypothesis H_0 that there are no sources; i.e. that all of the eigenvalues are zero. This hypothesis is accepted if the test statistic L is below a preset threshold. Otherwise H_0 is rejected, the largest eigenvalue is removed, and the test is repeated on the reduced set of eigenvalues. The test terminates if either H_0 is accepted at some stage (normal exit), or if there are no more eigenvalues to be tested. If the exit is normal the estimated number of sources is the number of iterations prior to the final acceptance of H_0. This is a modification of the Bartlett-Lawley procedure [9,10]. Its main disadvantage is that it is difficult to set the threshold for a desired test size.

Wax [11] has proposed two methods in which the test statistic is combined with a measure of the degrees of freedom:

$$\text{AIC:}\quad L + p(2M - p) \qquad (6)$$

$$\text{MDL:}\quad L + (1/2 \log n)p(2M - p) \qquad (7)$$

These are calculated recursively for p = 0,1,2,... as before, and the value of p for which the expression (6) or (7) reaches a minimum is taken as the estimated value. Wax has shown that expression (7) is asymptotically unbiased; however for the relatively small number of "snapshots" that we have used in simulations of the procedure we have found that expression (6) works better.

4. RECTIFICATION

In Fig. 1 we show a typical set of eigenvalues obtained in simulations. The eigenvalues are arranged in ascending order. One set corresponds to no signal, (SNR = 0). and the other set corresponds to a single signal with SNR = 1 present. The shape of the spectrum of ordered eigenvalues reflects the probability density of these eigenvalues.

If one assumes the eigenvalues under the noise-only hypothesis to be independently and identically distributed random variables with probability distribution F(x) and probability density f(x) then it can be shown [12] that the pdf of the ordered eigenvalues is given by:

$$f_k(x) = \frac{n!}{(k-1)!\ (n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x) \qquad (8)$$

We have calculated the expected value Ex_k of the ordered eigenvalues under the assumption that they are i.i.d. normal with zero mean and variance σ^2. A typical result of such a calculation is shown in Fig. 2 together with the expected eigenvalues of the zero-signal covariance matrix obtained by computer simulation. We see that the two ordered sets are very similar.

Subtracting the x_k from the λ_k results in a set of numbers having a much smaller standard deviation than the original set of eigenvalue estimates. Typical reductions in standard deviation of greater than a factor of five are easily achieved. The sensitivity of the likelihood-ratio test using these "rectified" eigenvalues is similarly increased.

Rectification clearly requires knowledge of the noise level. . It therefore compromises to some extent the feature of covariance-matrix subtraction of requiring no knowledge of the noise statistics. An estimate of the noise level can, however, generally be obtained. A possible method is to use a beamformer (which is just a particular way of processing the observations) and to use the smallest power level from the beamformer as an estimate of the noise level.

We have applied the likelihood-ratio test of Eq.(6)--i.e. the A.I.C. form--to simulated data. Typical results are presented in Figure 3. The plots shown in this figure are of the test function L + p(2M-p) for M = 11, p = 2, and signal-to-noise ratio of .7. In Fig. 3(a) we show the test function using only the positive half of the eigenspectrum. The companion result for the negative half eigenspectrum (not shown) is similar, but in the particular case illustrated shows no minimum at the correct value of p. Fig. 3(b) shows the test function applied to the absolute values of the eigenvalues. In both figures the solid line shows the test function applied to the estimated eigenspectrum directly; the dashed line shows the effect of first subtracting the expected mean of the ordered eigenvalues from Eq. (8). The figure shows the distinct performance improvement achieved by subtraction of the mean. Also it indicates that a test based on the absolute values of the entire eigenspectrum is preferable to one using only half of the eigenspectrum. This result has generally been corroborated by other simulation results.

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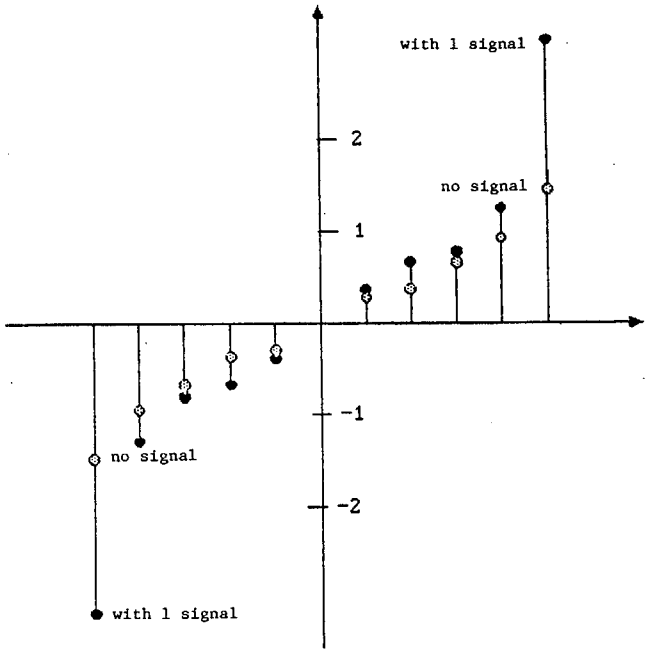

Figure 1. Typical Eigenspectra of $K_1 - K_2$

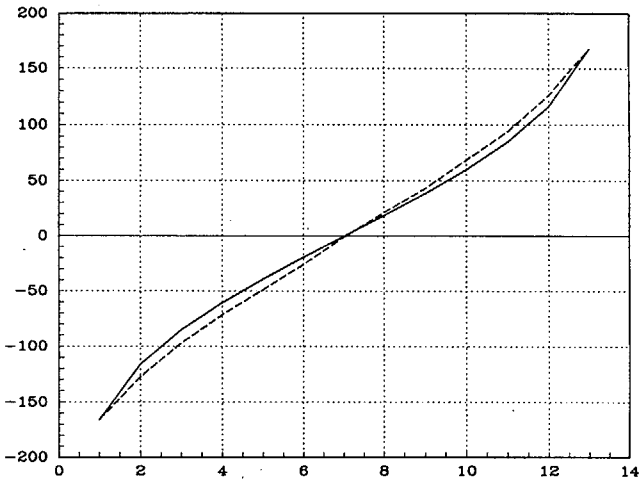

Figure 2 Sequence of Ordered Eigenvalues; (a) from Eq. (8) (solid); (b) from simulations (dashed).

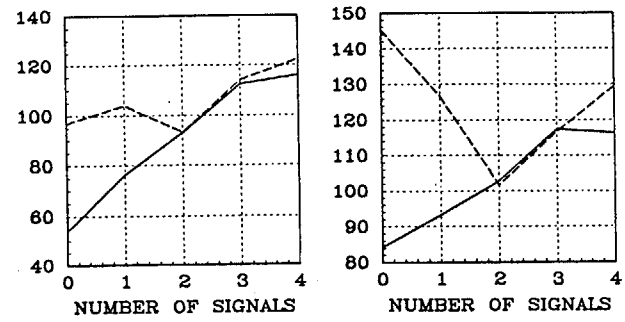

Figure 3. Typical simulation results: The test function L - p(2M - p). M=11, p=2, SNR = .7. (a) positive half of the eigenspectrum; (b) absolute value of the eigenspectrum; solid lines: without subtracting expected values; dashed line: with subtraction of the expected values.