ALGORITHM STRUCTURES FOR CYCLIC PHASE ESTIMATION[1]

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RESUMÉ

On étudie des estimateurs de phase rapides en utilisant la méthode presentée en [1]. L'idée centrale est la représentation du soit disant facteur d'observation par un train approprié de fonctions de Gauss. Le démodulateur qui en resulte a une structure très simple vis à vis des techniques alternatives pour la mise en oeuvre de filtres non linéaires. La performance, mésurée par l'erreur carrée moyenne est comparable à celles du filtre de masses ponctuelles [2] et du filtre de Fourier [3]. De l'autre côté, il n'a pas les problèmes associés au filtre de Fourier, à savoir: 1) la negativitée introduite quand on tronque la série; 2) le besoin de calculer des fonctions de Bessel; et, 3) la sensibilitée aux variations du rapport signal-bruit, ce que se traduit par la croissance du nombre de coéfficients de Fourier quand ce rapport là devient plus fort [4].

SUMMARY

The paper studies fast cyclic phase demodulators. They are derived by using the method presented in [1]. The central idea involves the representation of the so called sensor factor by a suitable train of Gaussian functions. The resulting demodulator is of a very simple structure when compared with alternative techniques for the implementation of the optimal nonlinear filter. The mean square performance compares to the ones of the point mass filter of [2] and of the Fourier filter of [3]. However, it does not exhibit the drawbacks associated with the Fourier filter, namely, i) the considerable negativity introduced by the truncated series, ii) the requirement of complex arithmetic and Bessel function evaluation, and iii) the unconsistent behavior for high signal to noise ratio, where the Fourier filter requires a larger number of Fourier terms to be retained [4].

ALGORITHM STRUCTURES FOR CYCLIC PHASE ESTIMATION

I - INTRODUCTION

A common practice in nonlinear estima-
tion linearizes the model nonlinearities about
the last estimate. By assuming a Gaussian a
priori probability density function, the re-
sulting structure is the Extended Kalman-Bucy
Filter. The Phase Locked Loop (PLL) is an
example of such a class of filters applied to
the phase estimation problem. If the mean
square estimation error is kept below the typ-
ical value of 0.25 rad^2, this device can be
considered a near optimal estimator. Above
that threshold there is a performance degrada-
tion. It is due to the fact that, for larger
errors (i.e. for stronger noise conditions),
the probability density function of the phase
process conditioned on the observations can
no longer be considered Gaussian. The density
becomes multimodal, being impossible to take
this information into account by linearizing
techniques.

If optimal nonlinear filtering is car-
ried out, by implementing the Bayes's Law,
better performance is expected. Previous stud-
ies have corroborated, by means of Monte Carlo
simulation, the preceding assertion. Direct
implementation of Bayes's Law leads however
to estimation structures that are more complex
than the PLL. This complexity, requiring a
larger computational effort, is the tradeoff
payed for the performance improvement. As will
be shown, the scheme proposed buys the per-
formance gains without the computational penal-
ties of more complex structures.

II - MODEL

A discrete scalar Brownian motion is
taken as the phase process.

$$x_{n+1} = x_n + u_n$$
$$x_1 = c \quad , \tag{1}$$

where $\{u_n\}$ is a white Gaussian sequence of
variance q, which is related to the parameter
q_c of the continuous corresponding process by
$q = \Delta q_c$. The sampling interval Δ must be suf-
ficiently small in order to guarantee that
the continuous and the discrete models describe
essentially the same phenomenum. The initial
condition x_1 is a random variable independent

of $\{u_n\}$ and has probability density function
$p(c)$.

The observation model is the vector
stochastic difference equation

$$\begin{bmatrix} z_{1n} \\ z_{2n} \end{bmatrix} = \begin{bmatrix} \cos x_n \\ \sin x_n \end{bmatrix} + \begin{bmatrix} v_{1n} \\ v_{2n} \end{bmatrix} \quad , \tag{2}$$

where $\{v_{1n}\}$ and $\{v_{2n}\}$ are two mutually inde-
pendent white Gaussian sequences of equal
variance r. They are also assumed to be inde-
pendent of $\{u_n\}$ and c. The variance r is re-
lated to the equivalent continuous variance
r_c by $r = r_c/\Delta$.

III - FILTER ALGORITHM

The problem consists on constructing an
estimate of x_n by processing the set of obser-
vations $Z_n = \{z_k, 1 \leqslant k \leqslant n\}$. The estimate is
defined by choosing a suitable cost function
$L(x-\hat{x})$. The solution requires, at each time
step, knowledge of the conditional density
function $p(x_n/Z_n)$. We shall denote $p(x_n/Z_n)$
by F(n) and call it the filter density.

The main task lies in the propagation
of F(n). This is recursively accomplished by
implementing the equations

Prediction $P(n+1) = S(n) * F(n)$ $\qquad$ (3)
Filtering $F(n) = C(n) H(n) \cdot P(n)$ $\qquad$ (4)

The symbol * in (3) denotes convolution.
The convolution kernel S(n), depending exclu-
sively on the phase dynamics, apart a nor-
malizing constant, is

$$S(n) = \exp\left[-\frac{1}{2q}(x_{n+1} - x_n)^2\right] . \tag{5}$$

The result of convolution, P(n+1), is the
predictor density. The symbol . in (4) repre-
sents a pointwise multiplication of H(n) by
P(n). The function H(n) is called the sensor
factor. It takes into account the observation
structure, being given by the expression

$$H(n) = \exp\left[(z_{1n} \cos x + z_{2n} \sin x)/r\right] . \tag{6}$$

The term C(n) in (6) is a normalization con-
stant. The a priori knowledge about the phase
process is given by $P(1) = p(c)$.

In order to implement (3) and (4) in a
digital computer a suitable representation

of the densities has to be used. In $[1]$, a Gaussian sum description of H(n) has been proposed. The procedure is as follows:

1. Sensor Factor Representation

Represent H(n) as an infinite set of Gaussian terms given by

$$\tilde{H}(n) = \sum_{i=-\infty}^{+\infty} \exp\left[-\frac{1}{2\sigma_n^H}(x-\eta_{in}^H)^2\right] \quad . \qquad (7)$$

The means η_{in}^H are computed according to

$$\eta_{in}^H = \eta_{0n}^H + 2\pi i \quad i=\ldots,-1,0,+1,\ldots \qquad (8)$$

where η_{0n}^H, the abcissa of the maximum of H(n) in the interval $[-\pi,+\pi)$, is given by

$$\eta_{0n}^H = \mathrm{tg}^{-1}\frac{z_{2n}}{z_{1n}} \quad . \qquad (9)$$

The variance σ_n^H is obtained by fitting one Gaussian function to each period of H(n) as shown in fig. 1. Its value is

$$\sigma_n^H = \pi^2 r/8 \sqrt{z_{1n}^2 + z_{2n}^2} \quad . \qquad (10)$$

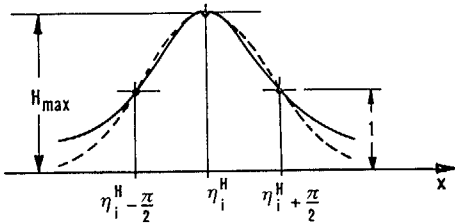FIG. 1 - Fitting one Gaussian function (---) to each period of the sensor factor (——)

As a consequence of this representation of H(n) and from the fact that S(n) is Gaussian, all the densities will be Gaussian sums, provided P(1), the initial condition, has the same form.

2. Filtering

2.1 - Multiplication

Assume P(n) is given by

$$P(n) = \sum_{i=1}^{N_n^P} K_{in}^P \exp\left[-\frac{1}{2\sigma_n^P}(x-\eta_{in}^P)^2\right] \quad , \qquad (11)$$

where K_{in}^P are the predictor weighting factors, σ_n^P is the predictor common variance and η_{in}^P

are the predictor means.

In order to build a finite representation of F(n), each term of P(n) multiplies only the J nearest terms of $\tilde{H}(n)$. The integer J, herein referred to as the multiplication parameter, is adjusted experimentally. The result of multiplication is

$$F(n) = \sum_{i=1}^{N_n^F} K_{in}^F \exp\left[-\frac{1}{2\sigma_n^F}(x-\eta_{in}^F)^2\right] \quad , \qquad (12)$$

with

$$\sigma_n^F = \frac{\sigma_n^P \sigma_n^H}{\sigma_n^P + \sigma_n^H} \quad , \qquad (13)$$

$$\eta_{in}^F = \eta_{\ell n}^P + \frac{\sigma_n^P}{\sigma_n^P + \sigma_n^H}(\eta_{\ell n}^P - \eta_{\ell_j n}^H) \quad , \qquad (14)$$

$$K_{in}^F \propto K_{\ell n}^P \exp\left[-\frac{1}{2(\sigma_n^P + \sigma_n^H)}(\eta_{\ell n}^P - \eta_{\ell_j n}^H)^2\right] \quad . \qquad (15)$$

$$i = J(\ell-1) + j$$

$$\ell = 1,\ldots, N_n^P$$

$$j = 1,\ldots,J$$

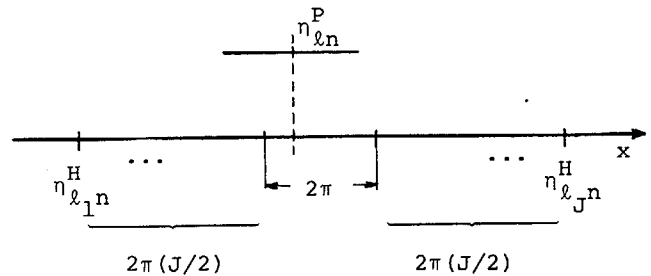The meaning of $\eta_{\ell_j n}^H$ is explained in fig. 2.

FIG. 2 - Explaining the meaning of $\eta_{\ell_j n}^H$.

2.2 - Agglutination

After multiplication, some of the modes of F(n) fall very near each other. When $|\eta_{in}^F - \eta_{jn}^F| < \beta_1$, the modes i and j are agglutinated. The resulting mean is the weighted average, the weighting factor being the sum of the weights. The variance remains unchanged. The threshold β_1, referred to as the agglutination parameter, is adjusted experimentally.

2.3-Elimination

Modes of $F(n)$ such that $K_{in}^F < \beta_2$ are elimi_nated. The constant β_2 is also set by simulation and is designated by the elimination parameter.

The elimination and agglutination procedures lead to a final number of terms of $F(n)$

$$N_n^F \leqslant N_n^P J \quad . \qquad (16)$$

The number N_n^F is called the filter dimension.

3 - Prediction

Convolution of $F(n)$ by $S(n)$ leads to the predictor density

$$P(n+1) = \sum_{i=1}^{N_{n+1}^P} K_{i(n+1)}^P \exp\left[-\frac{1}{2\sigma_{n+1}^P}(x - \eta_{i(n+1)}^P)^2\right] , \qquad (17)$$

where

$$N_{n+1}^P = N_n^F \quad , \qquad (17.a)$$

$$K_{i(n+1)}^P = K_{in}^F \quad , \qquad (17.b)$$

$$\eta_{i(n+1)}^P = \eta_{i(n+1)}^P \quad , \qquad (17.c)$$

$$\sigma_{n+1}^P = \sigma_{n+1}^F + q \quad . \qquad (17.d)$$

Formulae (16) and (17.a) indicate the possible change of the filter dimension at each iteration - an increase or a decrease.

4 - Estimation Criterion

Our interest lies on the cyclic phase demodulation. The loss function is periodic. The function $L(x_n - \hat{x}_n) = 2[1 - \cos(x_n - \hat{x}_n)]$ is a suitable choice [2].

By minimizing the conditional espectation

$$E\left[L(x_n - \hat{x}_n)/Z_n\right] = \int_{-\infty}^{+\infty} L(x_n - \hat{x}_n) F(n) dx_n \qquad (18)$$

with $F(n)$ given by (14), one is conducted to the optimal estimate

$$\hat{x}_n = tg^{-1} \frac{\sum_{i=1}^{N_n^F} K_{in}^F \sin \eta_{in}^F}{\sum_{i=1}^{N_n^F} K_{in}^F \cos \eta_{in}^F} \qquad (19)$$

IV - EXPERIMENTAL EVALUATION

The filter behavior is to be observed for different noise conditions.

The filter performance is quantified in terms of the mean square estimation error modulo 2π. The noise condition is measured in terms of the theoretical performance that would be achieved if the observations were linear. This performance is the error variance in steady state, R, which is supplied by the Ricati equation corresponding to the linearized continuous model. It is computed by

$$R = \sqrt{r_c q_c} \quad .$$

The linearized filter time constant τ is

$$\tau = \sqrt{r_c / q_c} \quad .$$

The sampling interval Δ, on the basis of which the discrete model is built, is such that $\Delta = 0.1\tau$, see [2] for details.

For a given value of R, the mean square errors modulo 2π, $\overline{\varepsilon_M^2}$, are computed by running 500 sample functions, each one with 130 points. The first 30 points of each run are discarded in order to measure only the steady state errors.

The value of $\overline{\varepsilon_M^2}$ is obtained through (20) and (21)

$$\overline{\varepsilon_n^2} = \frac{1}{500} \sum_{i=1}^{500} \left[(x_{in} - \hat{x}_{in}) \text{ modulo } 2\pi\right]^2 \qquad (20)$$

$$\overline{\varepsilon_M^2} = \frac{1}{100} \sum_{n=31}^{130} \overline{\varepsilon_n^2} \quad . \qquad (21)$$

The PLL and the NLF are evaluated simultaneously and start with the same initial condition.

1. A Simulation Result

Fig. 3 summarizes the results obtained for R= 0(dB) and the following filter parameters:

$$J = 2, \quad \beta_1 = 0.25, \quad \beta_2 = 0.001 \quad .$$

The evolutions of $\overline{\varepsilon_n^2}$ (PLL) and $\overline{\varepsilon_n^2}$ (NLF) are plotted together with the average value of the filter dimension for each value of n,

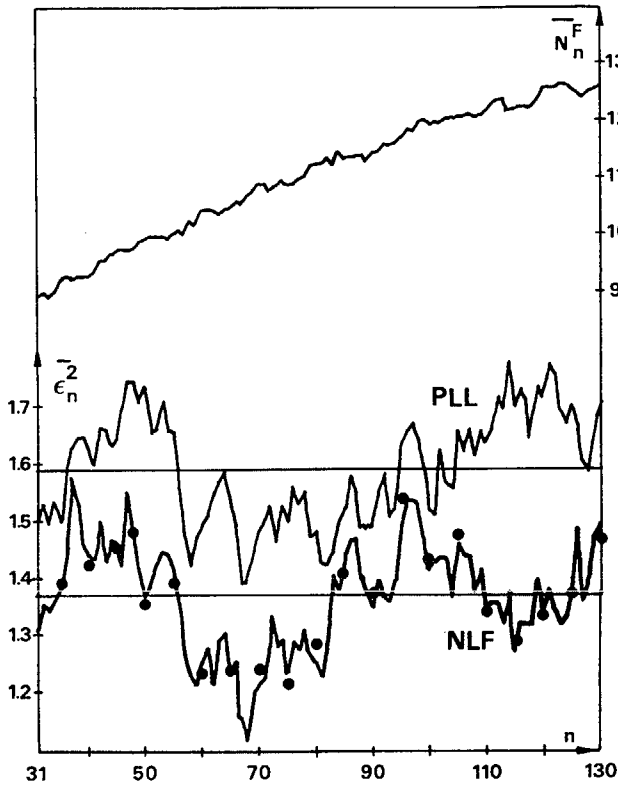$$\overline{N_n^F} = \frac{1}{500} \sum_{i=1}^{500} N_{in}^F \tag{22}$$

FIG. 3 - Simulation results for the PLL and the NLF, corresponding to R= 0 dB.

The horizontal lines represent the time averages of $\overline{\varepsilon_n^2}$ (PLL) and $\overline{\varepsilon_n^2}$ (NLF) as given by (21). Their values are 1.592 and 1.369 respectively.

Defining the performance improvement of the NLF over the PLL, in dB, as

$$\Delta\overline{\varepsilon_M^2} = 10 \log_{10} \overline{\varepsilon_M^2}(\text{PLL}) - 10 \log_{10}\overline{\varepsilon_M^2}(\text{NLF})$$

one computes $\Delta\overline{\varepsilon_M^2} = 0.65$ dB, which is of the same order of magnitude as the ones reported in [2].

Since the error process is a diffusion and the NLF algorithm allows the expansion of F(n) on the real line, the filter dimension N_{in}^F increases with time as can be seen by the plot of $\overline{N_n^F}$ in fig. 3.

This is unsuitable for practical reasons: the error performance is good but the filter exhibits an increasing complexity. A simplification is required to keep the filter

dimension constant and small.

2. Algorithm Simplification. The Matching Technique.

It was observed by Youssef in [5], by using a Fourier series expansion as a representation of H(n), that F(n) could be replaced, at each iteration, by only one Gaussian function, without significant loss of performance.

Let us denote this new representation of F(n) by

$$\tilde{F}(n) = \frac{1}{\sqrt{2\pi}\sigma_n^{\tilde{F}}} \exp\left[-\frac{1}{2\sigma_n^{\tilde{F}}}(x-\eta_n^{\tilde{F}})^2\right]. \tag{23}$$

According to [4], the parameters of $\tilde{F}(n)$ will be such that $\tilde{F}(n)$ and F(n) have the same first and second optimum moments relative to the same error criterion (the cyclic loss function). This condition corresponds to the matching equation (24).

$$\int_{-\infty}^{+\infty} e^{jmx}\left[F(n) - \tilde{F}(n)\right]dx = 0 \quad . \tag{24}$$

The new algorithm is now described:

A - The sensor factor representation is the same as in the preceding algorithm.

B - The predictor density P(n+1), obtained by convolution of two Gaussian functions S(n) and $\tilde{F}(n)$, is now Gaussian, with parameters

$$\sigma_{n+1}^P = \sigma_n^{\tilde{F}} + q \tag{25.a}$$

$$\eta_{(n+1)}^P = \eta_n^{\tilde{F}} \tag{25-b}$$

C - When multiplying P(n) by $\tilde{H}(n)$ only the J nearest terms of $\tilde{H}(n)$ relative to the unique term of P(n) are considered. This leads to a sum of J Gaussian terms for F(n).

D - Matching - By application of the matching equation (24) one is conducted to

$$\eta_n^{\tilde{F}} = tg^{-1} \frac{\alpha_s}{\alpha_c} \quad , \tag{26}$$

$$\sigma_n^{\tilde{F}} = \sigma_n^F - \ln(\alpha_s^2 + \alpha_c^2) \quad , \tag{27}$$

where

$$\alpha_s = \sum_{i=1}^{J} K_{in}^F \sin \eta_{in}^F \quad , \tag{28-a}$$

$$\alpha_c = \sum_{i=1}^{J} K_{in}^F \cos \eta_{in}^F \quad . \tag{28-b}$$

The cyclic estimate is now

$$\hat{x}_n = \eta_n^{\tilde{F}} \qquad (29)$$

Since $\eta_n^{\tilde{F}}$ belongs to $[-\pi, \pi)$ and $\eta_{n+1}^P = \eta_n^{\tilde{F}}$, the modes of $\tilde{H}(n)$ considered in the multiplication are the one in $[-\pi, \pi)$ and the adjacent ones. Next experiments correspond to J=3. Fig. 4 is a block diagram of the simplified (SNLF) algorithm just described.
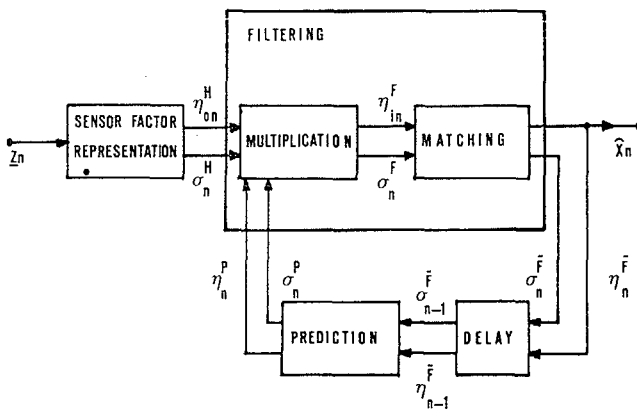

FIG. 4 - SNLF Block Diagram

V - SNFL PERFORMANCE EVALUATION

In order to see how this new filter (SNLF) compares with the preceding one (NLF), the same noise testing sequences (R= 0 dB) were used. A good agreement between $\overline{\varepsilon_n^2}$ (NLF) and $\overline{\varepsilon_n^2}$ (SNLF) was observed. The dot points in fig. 3 represent only the values of $\overline{\varepsilon_n^2}$ (SNLF) for n= 5m, m= 7,..., 26. The final mean square error was $\overline{\varepsilon_M^2}$ (SNLF)= 1.372 which is approximately the value of $\overline{\varepsilon_M^2}$(NLF). Thus we conclude that the same performance is achieved by setting J= 3 in the SNLF.

The continuous lines of fig. 5 represent the theoretical modulo 2π error variances versus R, for the PLL and the ideal linear filter. The upper dot points are the simulation results for the PLL; the lower ones correspond to the SNLF. Their respective values can be seen in table 1. One can see that: 1) the values of $\Delta \overline{\varepsilon_M^2}$, for R= -1,0,1, are of the same order of magnitude of the ones reported in [2]; 2) $\Delta \overline{\varepsilon_M^2}$ is consistently positive for all values of R, tending to zero as R approaches the region of linear behavior,

the dimension keeping constant. Point 2) emphasizes the difference relative to the Fourier filter which requires an increasing dimensionality as the SNR increases.
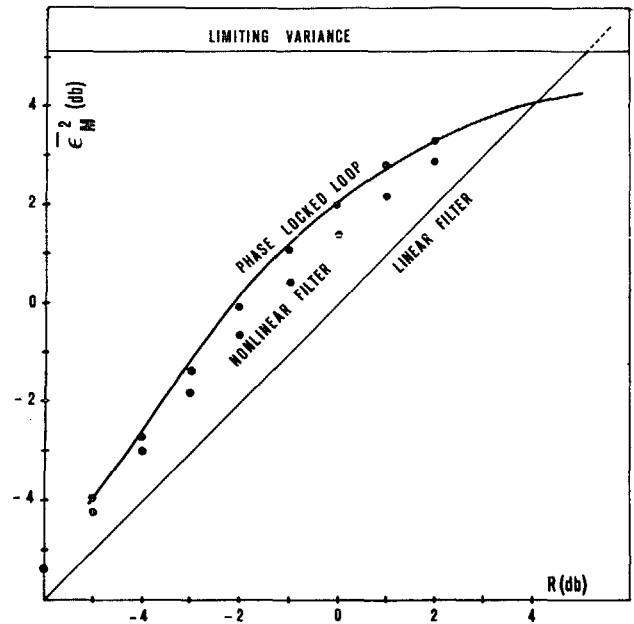

FIG. 5 - Simulation results (PLL, SNLF).

Table 1 - Mean square errors (modulo 2π),(dB)

R(dB)	$\overline{\varepsilon_M^2}$(P.L.L.)	$\overline{\varepsilon_M^2}$(N.L.F.)	Improvement $\Delta\overline{\varepsilon_M^2}$
- 6	- 5.371	- 5.402	0.031
- 5	- 3.920	- 4.240	0.320
- 4	- 2.711	- 3.026	0.315
- 3	- 1.398	- 1.825	0.427
- 2	- 0.105	- 0.644	0.539
- 1	1.071	0.417	0.654
0	2.018	1.372	0.646
1	2.823	2.194	0.629
2	3.296	2.885	0.401

The computation effort associated with the SNLF is,loosely speaking, equivalent to: 1) three Kalman-Bucy filter steps operating in parallel, the associated weighting factors involving three exponentiations; 2) one matching operation - equations(26),(27) and(28-a,b)- and 3) a Kalman-Bucy predictor step.

VI - CONCLUSION

The paper studies a cyclic phase demodulator designed by application of the optimal nonlinear filtering techniques. The crucial step lies on a representation of the sensor factor, whereby each of its periods is adjusted by a single Gaussian. The resulting filter's performance compares to the one of the point mass filter [2]. Since this filter propagates at each step a line density, the number of Gaussian parameters increases with time. To obtain a further simplified algorithm, an adaptation of the filtered density by a single Gaussian function is carried out at each step. The simplified nonlinear filter exhibits no noticeable loss of performance, nor any of the difficulties reported for the Fourier filter [4].

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