HUITIEME COLLOQUE SUR LE TRAITEMENT DU
SIGNAL ET SES APPLICATIONS

NICE du 1er au 5 JUIN 1981

A STOCHASTIC MODEL FOR SIGNAL PROCESSING AND PROPAGATION

George Adomian

Center for Applied Mathematics; University of Georgia; Tucker Hall; Athens, Georgia 30602, U.S.A.

RESUME

Le traitement de signaux, de modèles de clutter pour radar, propagation d'onde en acoustique, électromagnétisme et géophysique peuvent utiliser une méthodologie mettant en jeu la solution de systèmes dynamiques modelés par des équations de fonctionnement qui sont de nature assez générale et qui permettent à la fois un comportement stochastique et non-linéaire en utilisant la technique des fonctions de Green stochastique pour fournir la statistique du second ordre de la sortie du système. Ce travail évite toutes restrictions sur la nature et la grandeur des phénomènes aléatoires et non-linéaires et il est un outil puissant applicable là où une modélisation plus réaliste est nécessaire.

SUMMARY

The processing of signals, clutter models for radar, wave propagation in acoustic, electromagnetic, or geophysical cases all may be substantially aided by a methodology involving the solution of dynamical systems modelled by operator equations which are quite general in nature allowing both stochastic and nonlinear behavior using the stochastic Green's function technique to provide statistics to second order of the system output. This work avoids restrictions on the nature and magnitude of randomness and nonlinearity and offers a significant potential for applications where more realistic modeling is required.

A STOCHASTIC MODEL FOR SIGNAL PROCESSING AND PROPAGATION

Problems of propagation in a randomly varying channel or medium or the detection of signals in a reverberation or clutter limited, randomly space and time varying transmission media can be treated with a fundamental and unified approach through solution of the appropriate stochastic linear or nonlinear operator equations which model the system dynamics. This approach, due to Adomian [1], is a *computable* approximation method which allows realistic modeling which neither minimizes nor avoids the inherent nonlinearities and stochasticity in the system as do other generally available methods.

No perturbation, averagings, closure approximations, artificial assumptions for statistical separability or of white noise processes, monochromatic approximations, or linearization is involved. These common restrictions on the nature and magnitude of the fluctuations and nonlinearities may be adequate in some problems but entirely inadequate in others.

Our procedures allow stochastic operators, including differential operators with stochastic process coefficients as well as stochastic initial conditions or driving terms.

Let's consider wave propagation in a stochastic medium governed by a linear wave equation with a stochastic coefficient, because of a randomly time varying index of refraction, and stochastic boundary conditions, and input term. First of all, if this equation is correctly modeled assuming stochasticity from the beginning, the result is not the same as with replacement of quantities, such as the velocity in the d'Alembertian operator, for a deterministic problem by a stochastic quantity. Secondly, since the solution process is stochastic, we require a statistical description. The channel also is stochastic and is described by a random Green's function, i.e., the kernel of the integral operator representing the solution. The most complete statistical description would, of course, be given by the hierarchy of multidimensional probability distributions. However, such a characterization presupposes more information than is practically available for realistic media and is not necessary for system design or analysis. Actually, very recent work of Adomian and Malakian [2], connecting with work of Kuznetsov, Stratonovich, and Tikhonov [3] makes it conceivable to go further. Usually, we require only expressions for second-order statistical measures such as covariance functions and coherence functions for the solution process in terms of appropriate given statistical measures of the transmission medium and driving or source term. The kernel of the integral expressing the desired relation between a desired statistical measure of the solution stochastic process in terms of the corresponding statistical measure of the source term is Adomian's "stochastic Green's function" and involves statistical measures of the coefficient processes and boundary conditions. Thus, this approach derives the governing stochastic ordinary or partial differential equations from fundamental physical principles, obtains a stochastic solution in series form by the author's decomposition process analogous to obtaining ordinary Green's functions as the response for an impulse from decomposition of the forcing function or inhomogeneous term of a differential equation. Finally, appropriate ensemble averages provide desired statistical measures in terms of stochastic Green's functions and statistical measures of the source. The formalism provides a unifying framework and approach to many problems.

Suppose now that $\mathcal{L}(\bar{r},t,\omega)$ represents a linear stochastic differential operator and

$$\mathcal{L}(\bar{r},t,\omega)y(\bar{r},t,\omega) = x(\bar{r},t,\omega)$$

where $t \in T$, $\bar{r} \in R^3$, $\omega \in \Omega$ on the probability space (Ω,F,μ). $x(\bar{r},t,\omega)$ is the source or driving term and is a stochastic process. $y(\bar{r},t,\omega)$ is the solution field, whether electromagnetic or acoustic, whose statistical measures such as the expectation $\langle y\rangle$ and correlation $R_y(t_1,t_2)$ are sought. The solution can be written

$$y(\bar{r},t,\omega) = \iint h(\bar{r},\bar{r}',t,\tau,\omega)x(\bar{r}',\tau,\omega)dv\,d\tau$$

in terms of the *random* Green's function h. The (time) correlation of $y(\bar{r},t,\omega)$ is

$$R_y(t_1,t_2) = \langle y(\bar{r},t_1,\omega)\overset{*}{y}(\bar{r},t_2,\omega)\rangle$$

or

$$\iiiint G(\bar{r},\bar{r}',\bar{r}'',t_1,t_2,\tau_1,\tau_2)R_x(\bar{r}',\bar{r}'',\tau_1,\tau_2)dv'dv''d\tau_1d\tau_2$$

in general, where G is the stochastic Green's function for this particular statistical measure. Implicit in the derivation is an assumption that the propagation medium and the source are uncorrelated which is physically reasonable in most applications and quite different from the unrealistic assumptions of statistical independence implicit in hierarchy methods which require an independence between medium and the solution process for statistical separability and consequent closure of an otherwise infinite system of equations.

Central to the design and analysis of optimum detectors and estimators is the concept of the scattering function discussed by Sibul [4] in terms of our stochastic Green's functions, by Kennedy [5] for randomly dispersive communication channels, by Tuteur [6] for underwater acoustic communication with reflections from rough sea surfaces, etc. Scattering functions are stochastic Green's functions and they need to be derived by systematic application of stochastic operator theory as Sibul has pointed out, as well as exploring connections with ambiguity functions and optimal detection. Clutter scattering functions have been based on rather simplistic models and are not physically realistic. The methodology for solutions of stochastic operator equations, linear or nonlinear, is basic to any such analyses if realistic solutions are to be obtained. The determination of coherence functions in stochastic media is the same problem.

The basic theory began with a research study at Hughes Aircraft (August, 1960, R.S. 278, Linear Stochastic Operators, G. Adomian) generalizing an earlier classified study of signal processing in a randomized radar channel and extended in following years, finally being generalized to the nonlinear case in 1976. The present work is applicable to an equation of the form

$$\mathcal{F}y = x$$

where x is a stochastic process and $\mathcal{F}$ is a nonlinear differential operator which may involve wide classes of nonlinearities - polynomial, exponential, trigonometric, and even wide classes of functions involving y and its derivatives. The solution $y = \mathcal{F}^{-1}x$ is found by assuming a decomposition of $\mathcal{F}^{-1}x$ into components to be calculated first writing the equation in terms of its basic linear and nonlinear deterministic and stochastic parts $Ly + \mathcal{R}y + Ny + \mathcal{M}y = x$ where L indicates the linear deterministic operator and may be the mean value of the entire linear part $L + \mathcal{R}$, or $\mathcal{L}$, if it is readily invertible that way. If not, it may be simply the highest ordered derivative. Once L^{-1} is found, one writes the Volterra equation

$$y = F^{-1}x = L^{-1}x - L^{-1}Ry - L^{-1}Ny - L^{-1}My.$$

A term ϕ added to $L^{-1}x$ includes the solution of the homogeneous equation $Ly = 0$ and terms resulting from the stochastic bilinear concomitant if an adjoint operation is used in the second term because R involves differentiation acting on y. There are now several adaptations of Adomian's basic iterative method appropriate to different problems. One of these is as follows. Assuming a parametrized decomposition of y into $\sum_{n=0}^{\infty} \lambda^n y_n$ or equivalently $\sum \lambda^n F_n^{-1} x$ and analyticity for the deterministic nonlinear function Ny or f(y), and stochastic analyticity for the stochastic nonlinear function My or g(y), we have

$$y = y_0 - \lambda L^{-1}R\sum \lambda^n F_n^{-1}x - \lambda L^{-1}\sum A_n\lambda^n - \lambda L^{-1}\sum B_n\lambda^n$$

from which after setting $\lambda = 1$

$$y = y_0 + y_1 + \cdots$$

or

$$y = F^{-1}x = F_0^{-1}x + \cdots$$

where the components are determinable from

$$y_1 = -L^{-1}RF_0^{-1}x - L^{-1}A_0 - L^{-1}B_0$$

$$y_2 = -L^{-1}RF_1^{-1}x - L^{-1}A_1 - L^{-1}B_1$$

etc., the A_n and B_n being determinable by implicit differentiations as follows:

$$A_0 = A_0(y_0) = f(y(\lambda))|_{\lambda=0} = f(y_0)$$

$$A_1 = A_1(y_0,y_1) = f'(y(\lambda))|_{\lambda=0} = (df/dy)(dy/d\lambda)|_{\lambda=0}$$

$$A_2 = A_2(y_0,y_1,y_2) = (1/2)\{(d^2f/dy^2)(dy/d\lambda)^2 + (df/dy)(d^2y/d\lambda^2)\}|_{\lambda=0}$$

$$A_3 = A_3(y_0,y_1,y_2,y_3) = (1/6)\{(d^3f/dy^3)(dy/d\lambda)^3 + 2(d^2f/dy^2)(dy/d\lambda)(d^2y/d\lambda^2) + (d^2f/dy^2)(d^2y/d\lambda^2)(dy/d\lambda) + (df/dy)(d^3y/d\lambda^3)\}|_{\lambda=0}$$

$$\vdots$$

The B_n are similarly determined for the function $My = g(y) = g(y(\lambda))$ and differ only in involving a stochastic process, e.g., in a term such as $y(y) = b(t,\omega)y^3$, $b(t,\omega)e^y$, etc. If $f(y) = y^2$ as an example,

$$A_0 = y_0^2$$

$$A_1 = 2y_0y_1$$

$$A_2 = y_1^2 + 2y_0y_2$$

$$A_3 = 2(y_1y_2 + y_0y_3)$$

$$\vdots$$

If $f(y) = e^y$

$$A_0 = e^{y_0}$$

$$A_1 = e^{y_0}y_1$$

$$A_2 = (1/2)e^{y_0}(y_1^2 + 2y_0)$$

$$A_3 = (1/6)e^{y_0}(y_1^3 + 6y_1y_2 + 6y_3)$$

$$\vdots$$

Because of the need to calculate nonlinearities involving derivatives of y as well as y of the general form $Ny = f(y,y',\ldots)$, much more efficient procedures have been developed and are being reported elsewhere. Since each term is calculable in terms of the preceding term, no question of statistical separability will arise and no closure is necessary. The solution of equations of the form

$$Fu = g$$

where $g = g(x,y,z,t,\omega)$ and F involves partial differential operations is carried out similarly, although the expressions are considerably more complicated, and is being reported in journals. Statistical measures are then obtainable. If the entire equation is deterministic, this latter step is of course not necessary but the solution can nevertheless be obtained.

In stochastic cases, one need not consider the usual Gaussian or white noise restrictions and model with realistic processes as needed. For a propagation channel with deterministic characteristics, the input signal to a receiver will be a linear deterministic transformation of the transmitted signal.

It is interesting to note the development of an input-output point of view in analysis relying on the Ito approach to "stochastic differential equations", long used as a model for dynamical systems perturbed by white noise. This equation is written $dy = f(t,y)dt + g(t,y)dz$ but is actually understood in the sense of an integral equation

$$y(t) = y(0) + \int_0^t f(s,y(s))ds + \int_0^t g(s,y(s))dz(s)$$

where the last integral with z being the Wiener process is the Ito integral. The nondifferentiability of the Wiener process, however, is a *mathematical* property (as McShane has pointed out) not a physical property and our objective in physical problems is a physical solution, not theorems. In the input-output point of view the Ito equation is regarded as a mapping taking the "input" z into the "output" y. However, our approach, available since 1961, is far less restrictive since it does not restrict the nature of the process and offers a computable solution for real physical systems involving stochastic parameters.

The work discussed rests on secure analytical foundations as well as being useful. The linear space $V_{x(t,\omega)}(R)$ spanned by the elements of a second order stochastic process $x(t,\omega)$, with given definitions of inner product, norm, and distance functionals, constitutes a Hilbert space $L_2(\Omega,R)$ and the statistical description in terms of expectation, correlation, etc., determines the underlying space structure for each $x(t,\omega)$.

In optical communication systems when initially coherent light from a (single mode gas laser) transmitter is repeatedly scattered by a very large number of particles before it reaches an optical detector, the light at the detector is essentially a Gaussian process and the communication channel or scattering medium is a stochastic operator on the input process and can be described by stochastic differential equations such as those we consider. Applications can reasonably be expected in optics in image formation, resolution in microscopy, radio astronomy, lasers, and radar measurement and processing. Remote sensing of stochastic media such as internal waves or a turbulent medium, radar clutter, or pollutant mixing is also a possibility.

The analysis of input signals and noise through various linear and nonlinear devices, when the input is not necessarily white noise or even Gaussian, is essential in performance analyses in radar, communications, and other areas. Thus our objective has been to analyze a "stochastic filter" which stochastically transforms a stochastic input x to an output y, the objective being the determination of the statistical measures of the transformed process $y = Hx$. If the system is described by a differential equation $Ly = x$ in which the forcing function x is the system input and L is a stochastic (differential) operator, i.e., one involving stochastic coefficients, we need again to calculate statistical measures of y. Thus whether the system or channel is described by a "filter" operation or a differential equation, (linear or nonlinear), the appropriate operator can be effectively inverted to provide a stochastic solution process or output from which the necessary statistical measures can be calculated.

References

1) Adomian, G., Stochastic Systems Analysis, Applied Stochastic Processes, Academic Press, 1980.

2) Adomian, G. and Malakian, K., Stochastic Analysis, *Int. J. of Math. Modeling, v.* 1, no. 3, 1980, pp. 211-235.

3) Kuznetsov, P. I., Stratonovich, R. L. and Tikhonov, V. I., Quasi-Moment Functions in the Theory of Random Processes, *Theory Probab. and Its Appl.* 5, 80-97 (1960).

4) Sibul, L. H., Optimum Detection and Estimation in Stochastic Transmission Media, Applied Stochastic Processes, (ed. by G. Adomian), Academic Press, 1980.

5) Kennedy, R. S., Fading Dispersive Communication Channels," Wiley-Interscience, New York (1969).

6) Tuteur, F. B., J. F. McDonald and H. Tung, "Second-Order Statistical Moments of a Surface Scatter Channel with Multiple Wave Directions and Dispersions," *IEEE Trans. on Communications, v.* Com-24, no. 8, pp. 820-831 (1976).

7) Adomian, G., Nonlinear Stochastic Differential Equations, *J. of Math. Analysis and Applic., v.* 55, no. 1, August 1976, 441-452.