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SEPTIEME COLLOQUE SUR LE TRAITEMENT DU SIGNAL ET SES APPLICATIONS

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# CONVOLUTION PRESERVING TRANSFORMATIONS

## A NEW INTERPRETATION

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## RESUME

Pour obtenir numériquement les spectres fréquentials à l'aide d'échantillage, on prend ordinairement les valeurs du signal intermediares entre les points de mesure être intepolée par la série de SHANNON. La transformation de FOURIER appliquée à un signal donnée par cette série conduit à la transformation discrète de FOURIER. Si c'est utilisée pour le calcul du spectre d'un signal, on obtient des erreurs quand le signal n'est pas à bande limite.

Nous montrons que c'est possible de circumvenir cette problème si l'on utilise la série infini de NEWTON-GREGORY à la place de la série de SHANNON. Cette série converge si c'est égal à la transformation de POISSON d'une série polynômiale de LAGUERRE. C'est une interpetation singulier à une transformation conservant le produit de convolution parce que la transformation de POISSON est utilisée d'obtenir une séquence des échantillons réels à la place d'une autre sorte d'un signal discret. Les signaux avec échantillons que on peut obtenir dans cette manière satisfont un théorème nouveau d'échantillonage. Avec cet théorème nouveau, c'est possible d'obtenir des spectres fréquentials plus précisement.

## SUMMARY

To compute spectra, it is usual to assume that the analog signal which a given sequence of sample values represents can be taken to be the bandlimited signal the SHANNON series fits through the samples. The FOURIER transform of this series gives the discrete FOURIER transform computing formula. This formula gives a computed spectrum which can be badly distorted by aliasing if the true analog signal is not well bandlimited.

We will show that this distortion can be averted if one uses the infinite NEWTON-GREGORY series instead of the SHANNON series to interpolate signal values between sampling instants. The NEWTON-GREGORY series converges when it can be generated by POISSON transforming a LAGUERRE polynomial series term by term. This gives an unusual interpretation to a convolution preserving transformation because it uses the POISSON transform to mathematically generate true signal samples, instead of to represent an analog signal by an alternative type of digital signal. The signals whose samples this convolution preserving transformation generates from LAGUERRE series obey a new sampling theorem; and when it applies, a signal can be specified by its samples using a NEWTON-GREGORY series whose FOURIER transform gives a computing formula yielding alias-free spectral values.

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## 1. - INTRODUCTION

C. BOZZO [1] has presented a survey describing a number of transformations which allow a function of a continuous variable to be transformed into a number sequence so that the operation of convolution is preserved. It is usual to view these transformations as mathematical devices for specifying digital signals other than periodic samples, with which analog filtering can be imaged by digital filtering. We will show that it can be useful to give a totally different interpretation to at least one convolution preserving transformation, and to regard it as a mathematical device for generating a true sequence of signal samples.

If a function can be represented as a series of LAGUERRE polynomials, then the POISSON transform of the series generates samples of an analog signal whose values between sampling instants are correctly interpolated by a convergent NEWTON-GREGORY series. This says that the infinite NEWTON-GREGORY series can interpolate analog signal values between sampling instants for signals whose samples can be generated by this convolution preserving transformation, much as the SHANNON series can do this for signals which have bandlimited spectra. Since the generating function must be representable as a LAGUERRE series, it should be possible to exploit the LAGUERRE transform as well as the POISSON transform to analyze the processing of samples of these analog signals.

To be useful, this representation must be applicable to signals encountered in applications of signal processing. The purpose of this paper is to show that it is applicable to an important class of signals. We will show that these signals are governed by a new sampling theorem which allows an unbandlimited signal to be specified by its samples if the poles of its LAPLACE transform all have frequencies below one-sixth the sampling rate. The FOURIER transform of the NEWTON-GREGORY series representation these signals have can be used to compute spectra without the aliasing distortion which usually occurs when signals are sampled at less than the NYQUIST rate.

## 2. - THE INFINITE NEWTON-GREGORY SERIES

C. LANCZOS [2] has shown that those positive-time signals whose samples can be generated by POISSON transforming a LAGUERRE series, and for which the infinite NEWTON-GREGORY series

$$f(t) = \sum_{k=0}^{\infty} \Delta^k f(0) \frac{t^{(k)}}{k!} \quad , \quad t \geq 0 \qquad (1)$$

gives a correct convergent interpolation through unit spaced samples $f(n)$ are uniquely characterizable by their being capable of being generated by a function $g(\alpha)$ satisfying the condition

$$\int_0^{\infty} |g(\alpha)| e^{-\alpha/2} d\alpha < \infty \qquad (2)$$

using the integral

$$f(t) = \int_0^{\infty} g(\alpha) \frac{\alpha^t}{t!} e^{-\alpha} d\alpha \quad , \quad t \geq 0. \qquad (3)$$

This last formula is the convolution preserving POISSON transform when $t$ is an integer. $\Delta^k f(0)$ denotes the k-th forward difference of $f(t)$ at $t = 0$ and is equal to the sum $\Delta^k f(0) = f(k) - f(k-1) + (k^{(2)}/2!) f(k-2) --- + (-1)^k f(0)$ of the first $k + 1$ signal samples weighted with alternating-signed binomial coefficients. $t^{(k)}$ denotes the k-th descending factorial power of $t$ and equals the product of the $k$ factors $t(t-1)(t-2)---(t-k+1)$.

To delineate types of signals which the infinite NEWTON-GREGORY series can interpolate correctly between sampling instants, note that the constraint (2) makes $g(\alpha)$ have a LAPLACE transform

$$G(s) = \int_0^{\infty} g(\alpha) e^{-s\alpha} d\alpha \qquad (4)$$

which converges when $Re(s) \geq 1/2$. It is straightforward to show that $G(s)$ is related to the z-transform $F(z)$ of the samples $f(n)$ which the POISSON transform specifies by

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} = G(s)\Big|_{s=1-z^{-1}} . \qquad (5)$$

This restricts the values of $z$ where $F(z)$ can have poles to those corresponding to values of $s$ where $G(s)$ can diverge and have poles. These are given by $z = 1/(1-s)$ with $Re(s) < 1/2$, and correspond to points inside a unit radius circle around the point $z = 1 + j0$ on the z-plane. For the pole at $z = \exp(\sigma)\exp(j\omega)$ of the z-transform $F(z)$ of a sampled signal $\exp(\sigma + j\omega)n$ to be within this circle, the signal must have a frequency $|\omega| < \pi/2$ radians/sampling interval and a decay factor $\exp(\sigma) < 2\cos(\omega)$.

Restricting $\omega$ and $\sigma$ in this way is equivalent to restricting $\exp(\sigma + j\omega)$ to equal $1 + \zeta$ where $\zeta$ is a complex number with magnitude $|\zeta| < 1$. $(1 + \zeta)^t$ is the signal which formula (3) yields when $g(\alpha) = (1 + \zeta)^{-1}\exp(\zeta/(1 + \zeta))\alpha$. This $g(\alpha)$ has the LAGUERRE polynomial series representation

$$\frac{1}{1+\zeta} e^{\frac{\zeta}{1+\zeta}\alpha} = \sum_{k=0}^{\infty} \zeta^k L_k(\alpha) \quad , \quad \alpha \geq 0 \qquad (6)$$

where $L_k(\alpha) = \Delta^k(\alpha^n/n!)|_{n=0}$. If this is transformed term by term using formula (3), one obtains the NEWTON-GREGORY series

$$(1 + \zeta)^t = \sum_{k=0}^{\infty} \zeta^k \frac{t^{(k)}}{k!} \quad , \quad t \geq 0. \qquad (7)$$

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When t is an integer, this is just the familiar binomial theorem. When t is not an integer, it is an infinite series equal to the TAYLOR series expansion of $(1 + \zeta)^t$ as a function of $\zeta$ and converges when $|\zeta| < 1$. As we have already noted, when $|\zeta| < 1$, the complex number $\exp(\sigma + j\omega) = 1 + \zeta$ has an angle $|\omega| < \pi/2$ and a magnitude $\exp(\sigma) < 2\cos(\omega)$. Then, the values of the signal $\exp(\sigma + j\omega)t$ are correctly interpolated between sampling instants by the NEWTON-GREGORY series (7). By differentiating (7) term by term with respect to $\zeta$, it can be shown that any signal of the form $t^n \exp(\sigma + j\omega)t$, $n \geq 0$ is interpolable by a NEWTON-GREGORY series when $\omega$ and $\sigma$ satisfy these inequalities.

## 3. - SAMPLING THEOREM

Although it is impossible to sample a signal forever, a signal $\exp(\sigma + j\omega)t$ with $\sigma < 0$ can be sampled until its amplitude has decreased to a negligible value. If $\sigma < 0$, the condition $\exp(\sigma) < 2\cos(\omega)$ is always satisfied when $|\omega| < \pi/3$ radians/sampling interval. Using the superposition principle and changing notation so as to explicitly include the sampling interval T, this gives the sampling theorem:

Any positive-time signal of the form

$$f(t) = \sum_{k=1}^{N} a_k t^{n_k} e^{\sigma_k t} \cos(\omega_k t + \theta_k) \quad , \qquad \sigma_k < 0 \qquad (8)$$

with $n_k \geq 0$ is uniquely specified by its values at sampling instants $t = nT$, $n = 0,1,2,\ldots$ if $\omega_k T < \pi/3$ for every k; or equivalently, if all the poles $s_k = \sigma_k + j\omega_k$ for its LAPLACE transform F(s) have frequencies $|f_k| = |\omega_k/2\pi|$ below 1/6 the sampling rate 1/T (or between 1/6 and 1/4 the sampling rate with $\sigma_k$ satisfying the tighter constraint $\exp(\sigma_k T) < 2\cos(\omega_k T)$).

## 4. - COMPUTATION OF FOURIER INTEGRAL

The NEWTON-GREGORY series, whose terms are not bandlimited and have non-zero FOURIER transforms both above and below the folding frequency, can be FOURIER transformed term by term.

In numerical computation one can only use the FOURIER transform of the signal given by a NEWTON-GREGORY series with a finite number of terms:

$$\sum_{k=0}^{n} \Delta^k f(0) \frac{t^{(k)}}{k!} \quad , \quad t \geq 0 \qquad (9)$$

fitting an n-th degree polynomial through the first n+1 samples. The fact that the n+1 coefficients $\Delta^k f(0)$ depend only on the first n+1 sample values, regardless of the value of n, makes them the correct first n+1 coefficients of the series for a signal which is only interpolable exactly by an infinite series. Nonetheless, the NEWTON-GREGORY series is only suited for interpolating signal values between sampling instants and not for extrapolating beyond the sampling range. Therefore, we FOURIER transform the truncated in time polynomial

$$\sum_{k=0}^{n} \Delta^k f(0) \frac{t^{(k)}}{k!} [u(t) - u(t-n)] \qquad (10)$$

instead of (9). Using STIRLING numbers $S(k,\ell)$, $t^{(k)}$ can be re-expressed in terms of ordinary powers of t as

$$t^{(k)} = \sum_{\ell=0}^{k} S(k,\ell) t^{\ell}. \qquad (11)$$

The formula can then be FOURIER transformed so as to obtain the computing formula:

$$\hat{F}(j\omega) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} \sum_{\ell=0}^{k} S(k,\ell) \frac{\ell!}{(j\omega)^{\ell+1}} \cdot \left[ 1 - e^{-j\omega n} \sum_{m=0}^{\ell} \frac{(j\omega n)^m}{m!} \right] \qquad (12)$$

for the approximation to the FOURIER transform $F(j\omega)$ of f(t) given by the FOURIER transform of a truncated polynomial fitted through n+1 signal values. This formula specifies an alias-free value of the spectrum of the truncated polynomial at any frequency.

Figures 1-4 show plots of spectral amplitude versus frequency in radians/sampling interval for signals truncated after the number of sampling intervals N noted in each figure. Figure 1 is for a parabolic pulse of

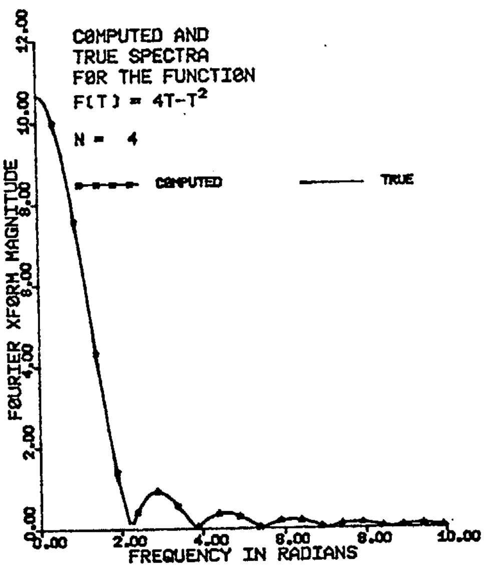


Fig. 1. Spectra of polynomial pulse.

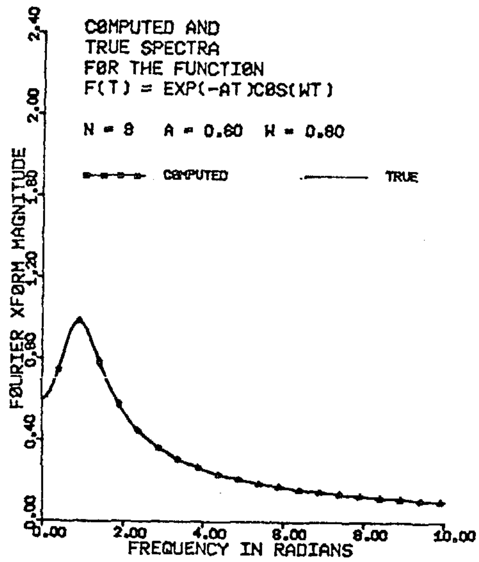


Fig. 2. Spectra of damped sine wave.

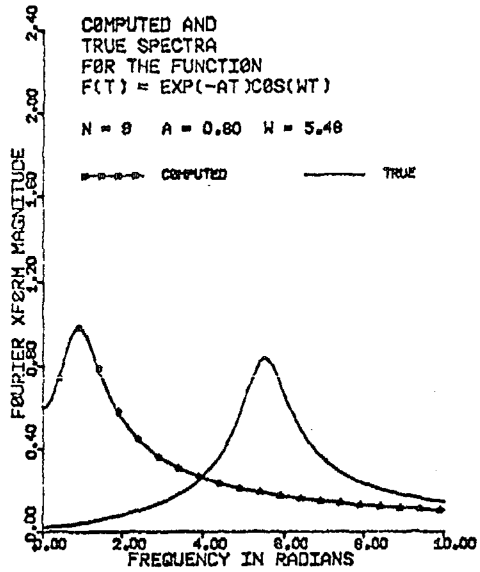


Fig. 4. Spectra of signal with an alias with $\omega < \pi/3$.

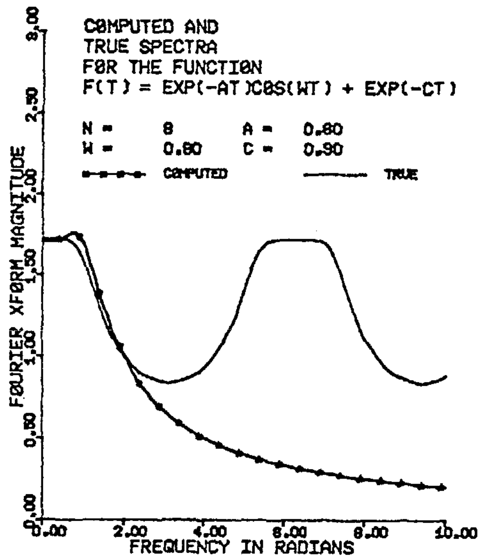


Fig. 3. Spectra of sum of damped sine wave and exponential.

finite duration exactly interpolable by a NEWTON-GREGORY series. The computed spectrum accurately reproduces the oscillations which the GIBBS phenomenon gives to the spectrum as a result of truncation. In the other figures, the true spectra shown are for untruncated signals. In Figs. 2 and 3, where the signal is not truncated until it has decayed to a small value, the true and computed spectra are indistinguishable. Fig. 3 also displays the usual periodic DFT spectrum computed for the same signal from the same set of samples. It has been scaled by a factor of .609 to make it agree with the true spectrum at $\omega=0$ and make it easier to see how its form is distorted by aliasing. The computed spectra in Figs. 2 and 4 are identical because Fig. 2 is for a lower frequency alias with the same set of sample values as the signal in Fig. 4 whose LAPLACE transform has a higher pole frequency than the sampling theorem allows.

## 5. - CONCLUSIONS

It has been shown that the infinite NEWTON-GREGORY series interpolates correct signal values between sampling instants for superpositions of damped sine waves when the poles of their LAPLACE transforms all have frequencies below 1/6 the sampling rate. Plots of computed spectra have been exhibited which show that it can be possible to numerically evaluate the spectrum of a signal satisfying this sampling criterion using the FOURIER transform of a few terms of a NEWTON-GREGORY series. By showing alias-free spectral values computed both above and below the folding frequency, these plots show that aliasing distortion is circumvented when spectra are computed in this way.

To obtain these results, we have not had to use the fact that the POISSON transform specifies samples of these signals so that the discrete convolution of two sample sequences is imaged by the continuous convolution of their generating functions. This makes the application of the mathematics of convolution preserving transformations [1] to extend the analysis presented here an inviting area for further research.

## ACKNOWLEDGEMENTS

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[2] C. LANCZOS "Linear Differential Operators", VAN NOSTRAND, New York 1961.